Infinite Wedding Cake

In this section we see another glimpse into this magical world of the infinite. It too will require a shift in our intuition.

To analyze this example we must first visit another famous infinite series:

$$\frac{1}{1}+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\dots$$

1. Write out the next ten terms in this infinite series.

2. What is the sum of the first five terms in this series?

3. What is the sum of the first ten terms in this series?

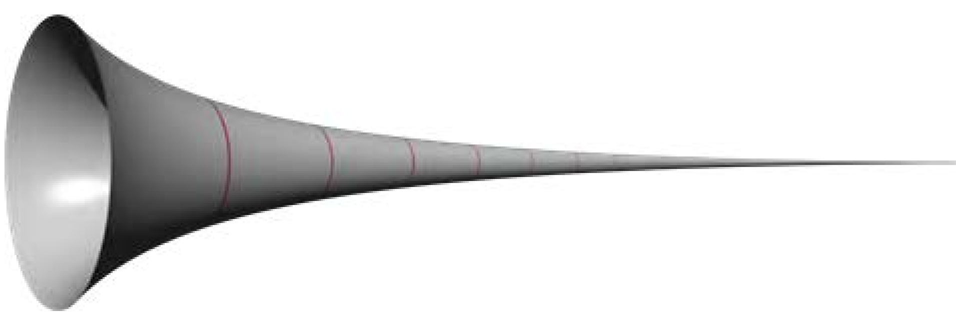
4. Show that your answers to the previous problems are close to $\frac{\pi^2}{6}$.

Showing, in 1734, that the sum of the infinite series $\frac{1}{1}+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\cdots$ is precisely equal to $\frac{\pi^2}{6}$ was one of the great early triumphs of Leonard Euler (Swiss mathematician, physicist, astronomer ;1707 - 1783), one of the three greatest mathematicians of all time. It is also interesting to note that over 200 years later the sums of many closely related infinite series remain unknown. In fact, we do not know the sums of any of the p-series with odd exponents greater than one:

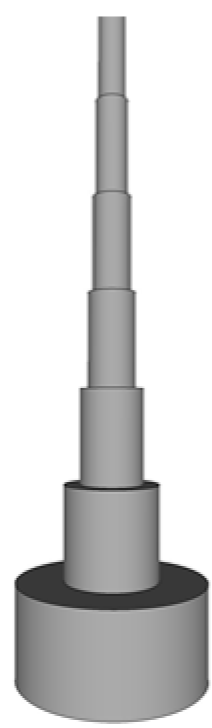
$$\frac{1}{1^3}+\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{4^3}+\dots$$
$$\frac{1}{1^5}+\frac{1}{2^5}+\frac{1}{3^5}+\frac{1}{4^5}+\dots$$
$$\frac{1}{1^7}+\frac{1}{2^7}+\frac{1}{3^7}+\frac{1}{4^7}+\dots$$
$$\vdots$$

Indeed, these sums are values of the Riemann Zeta Function which is the centerpiece of the **Riemann Hypothesis**. This unsolved problem is so important there is a $1,000,000 (US) prize for its solution as part of the Clay Mathematics Institute *Millenium Prize Problems* and it is the focus of many new books for general audiences.

The object below is called Gabriel's Horn. It was invented by Evangelista Toricelli (Italian Mathematician and Physicist; -) in 1641 and had important links to the forthcoming invention of the calculus and sparked significant philosophical debates.[1]



Gabriel's Horn.



Gabriel's Wedding Cake

[1] See "Gabriel's Wedding Cake" by Julian F. Fleron, The College Mathematics Jo L, January 1999, pp. 35 - 38 for more details and references.

The object on the right shares similarly surprising properties as Gabriel's Horn. It was invented by Julian F. Fleron (American Mathematician and Teacher; 1966 -) in 1998 and is called Gabriel's Wedding Cake.

Gabriel's Wedding Cake is constructed by piling an infinite number of cylindrical cake layers on top of one another. The height of each of these layers is 1, while the radii of these layers are $1, \frac{1}{2}, \frac{1}{3}, \cdots$, respectively.

5. **Classroom Discussion**: Can the volume and surface area of Gabriel's Horn and Wedding Cake be measured? Do you have predictions about the size of these measures?

Describe the formulae for the volume and surface area of a cylinder.
Find an expression, leaving πas a symbolic value without converting it to a decimal value, for the volume of …

6. ...the first layer of the cake.

7. ...the second layer of the cake.

8. ...the third layer of the cake.

9. Describe the pattern you see in the measures of the volumes in the previous investigations. Use it to determine expressions for the volumes of the next four layers of the cake.

10. Determine an infinite series that measures the volume of the entire cake.

11. Using a series that we have previously considered, find the exact volume of the cake.

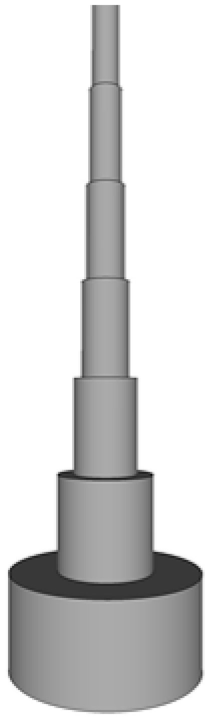
12. Find the total exposed area of the tops of the cakes. (Hint: it's easy if you think about all of them together, otherwise it is hard.)
Find an expression, leaving πas a symbolic value without converting it to a decimal value, for the surface area of the side of...

13. ...the first layer of the cake.

14. ...the second layer of the cake.

15. ...the third layer of the cake.

16. Describe the pattern you see in the measures of the surface areas in the previous investigations. Use it to determine expressions for the surface areas of the next four layers of the cake.



Gabriel's Wedding Cake

17. Use the previous investigation to determine an infinite series that represents the volume of the entire cake.

18. Using a series that you have previously considered, find the exact surface area of the cake.

19. Use the previous investigation to determine an infinite series that represents the volume of the entire cake.

20. Using a series that we have previously considered, find the exact surface area of the cake.

21. **Classroom Discussion**: These investigations prove that the Gabriel's Wedding Cake has a volume of just over 5 cubic units while it has an infinite surface area. Does this seem feasible? If we had a single cake pan to make this cake we could make the batter to fill it, but could we grease the surface of the pan (so the cake didn't stick)? This seems problematic, doesn't it? Are there ways out of this dilemma? What does this do to our understanding of the infinite?