Devilish Series

> The divergent series are the invention of the devil, and it is a shame to base on them any demonstration whatsoever. By using them, one may draw any conclusion he pleases and that is why these series have produced so many fallacies and so many paradoxes.
>
> **Neils Henrik Abel** (Norwegian Mathematician; 1802 - 1829)

There is no reason that infinite series must have all positive terms. For example, the alternating harmonic series is built from the harmonic series but with every other term negative:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots$$

Shown below are the results of adding a small number of terms of this series.

Partial Series	Partial Sum
$1 - \frac{1}{2} + \frac{1}{3} - \dots - \frac{1}{100}$	0.68817217931019520324
$1 - \frac{1}{2} + \frac{1}{3} - \dots - \frac{1}{500}$	0.69214818055794532542
$1 - \frac{1}{2} + \frac{1}{3} - \dots - \frac{1}{1000}$	0.69264743055982030967
$1 - \frac{1}{2} + \frac{1}{3} - \dots - \frac{1}{5000}$	0.69304719055994510942

For each of the following, sum the indicated number of terms to 9 decimal places:

1. ...the first four terms.

2. ...the first five terms.

3. ...the first six terms.

4. ...the first seven terms.

5. ...the first eight terms.

6. ...the first nine terms.

7. ...the first ten terms.

8. ...the first eleven terms.

9. ...the first twelve terms.

10. **Classroom Discussion**: Do the patterns in your partial sums allow you to determine whether a partial sum over- or under-estimates the actual sum? Can you determine a bound on how much your partial sums over- and under-estimate the actual sum? How can these observations be combined with your data to find the sum of the alternating harmonic series correct to many decimal places? Do this.

Natural Logs

The *natural logarithm* is one of the more important functions in mathematics. Its values can be approximated on most calculators - you use the ln function button. The value of $ln(2)$, correct to 9 decimal places, is 0.6931471806. That this is the sum of the alternating harmonic series.

11. Is 2 + 3 = 3 + 2? Why?

12. Is 219 + 3,427 + 5,392 = 5,392 + 219 + 3,427? Why?

13. Is 12 + 3,492 + 12,987 + 438 + 29,351 + 423 = 438 + 12 + 423 + 12,987 + 29,351 + 3,492? Why?

14. Does the sum of finitely many terms remain the same when the order of the terms are rearranged?

15. Do you think the sum of infinitely many terms remains the same when the order of the terms are rearranged?

When we rearrange the order in which we add terms in a series, finite or infinite, we call this a **rearrangement** - no surprise there. Let us investigate what happens when we rearrange the alternating harmonic series.

16. What are the terms in the next three sets of parenthesis in the series:

$$1-\frac{1}{2}+\left(\frac{1}{3}-\frac{1}{4}-\frac{1}{6}\right)+\left(\frac{1}{5}-\frac{1}{8}-\frac{1}{10}\right)+\ldots$$

17. Explain why the infinite series in Investigation 16 is a rearrangement of the alternating harmonic series.

18. Simplify and reduce the terms in each of the five sets of parenthesis in Investigation 16, leaving your results as fractions.

19. Show that each of the results in Investigation 18 is given by $\frac{-1}{4n(2n+1)}$ for appropriate values of n. (Note: With a little algebra, this result can be proved to continue indefinitely.)

20. What can you conclude about the sum of the infinite series in Investigation 75? Does this result agree with the sum of the series prior to its rearrangment?

Consider now the infinite series

$$1+\left(-\frac{1}{2}+\frac{1}{3}+\frac{1}{5}\right)+\left(-\frac{1}{4}+\frac{1}{7}+\frac{1}{9}\right)+\ldots$$

21. What would the terms in the next three sets of parenthesis in this series be?

22. Explain why this infinite series is a rearrangement of the alternating harmonic series.

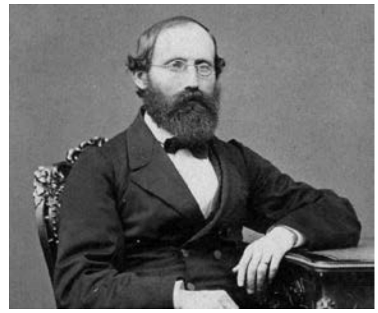
23. Simplify and reduce the terms in each of the five sets of parenthesis, leaving your results as fractions.

24. Show that each of the results is given by $\frac{-1}{2n(4n-1)(4n+1)}$.

25. What can you conclude about the sum of this infinite series? Does this result agree with the sum of the series prior to its rearrangment?

We have now shown that the alternating harmonic series has three different sums when it is rearranged! These are not fictitious sums like those of the Grandi series. These are real sums whose exact values can be rigorously determined. And this shock is just the beginning. For the alternating harmonic series can be rearranged to have any sum! This is the striking result of Georg Friedrich Bernhard Riemann (German mathematician; 1826 - 1866):

Riemann's Rearrangement Theorem: Suppose that an infinite series of both positive and negative terms converges while the series formed by making all of the original series' terms positive diverges. (Such a series is called **conditionally convergent**.) Let T be any real number, $+\infty$ or $-\infty$. Then there is a rearrangement of the series that converges to T.



This is a remarkable theorem which highlights the limits of our finite intuitions in dealing with the infinite. Unlike some of the earlier surprises and paradoxes, we are not seeing the faults in our reasoning. We've illustrated this result with rigor, the theorem itself is established deductively in a fairly straightforward way (see 1.12 Extension), and we must accept this result as a wonderful glimpse into the world of the infinite.