Divergent Series

So far, you have only explored infinite series that are well-behaved. What this means must now be determined, but, up to this point, all the infinite series have been convergent. Now you will work to clarify what it means for infinite series to be well-behaved. This is an absolutely critical matter, as realized by the famous, but tragically short-lived,[1] Neils Henrik Abel (Norwegian Mathematician; 1802 - 1829):

> If you disregard the simplest cases, there is in all of mathematics not a single infinite series whose sum has been rigorously determined. In other words, the most important parts of mathematics stand without a foundation.

1. Consider the sequence $(-1),(-1)^2,(-1)^3,(-1)^4,\ldots$ Simplify the terms in this sequence so the sequence can be written without using exponents.

2. Let $r = -1$. With the help of the ideas in the previous investigation, express the geometric series $1 + r + r^2 + r^3 + \ldots$ without the use of exponents.

The infinite series in Investigation 2 is called the **Grandi series** after Guido Grandi (Italian Mathematician and Jesuit Philosopher; 1671 - 1742).

3. What do you think the sum of the Grandi series is? Explain.

[1] Abel made brilliant and insightful contributions to modern mathematics, but, sadly, died of tuberculosis at the age of 27. He is a revered figure in Norway and the Abel Prize is one of the more significant awards in the field of mathematics.

4. Using parentheses, group the first two terms in the series together, the next two terms together, the next two terms after that together, etc. What value does this suggest the sum of the Grandi series will be?

5. Suppose now that you left the first term by itself and instead used parentheses to group the second and third terms together, the fourth and fifth terms together, etc. What value does this suggest the sum of the Grandi series will be?

6. Are your results in Investigations Investigation 4 and Investigation 5 compatible? Explain.

7. The situation in Investigation 6 was described by Grandi as “comparable to the mysteries of Christianity... paralleling the creation of the world out of nothing.” What do you think of Grandi’s description?

8. Because the Grandi series arose from a geometric series it seems reasonable to use our formula above to ascertain a value for the sum of the series. What value does the formula predict?

9. **Classroom Discussion**: What do these results suggest about the numbers 0, $\frac{1}{2}$, and 1? Is any of this reasonable? What do you think is an appropriate value for the sum of the Grandi series?

Harmonic Series

Even the most gifted mathematicians were troubled by paradoxes like the Grandi series. Leonard Euler (Swiss Mathematician; 1707 - 1783), one of the greatest mathematicians of all times, held firmly to the conviction that 1/2 was the proper sum of the Grandi series. The contemporary practice is to name infinite series for which an appropriate sum cannot be found as a *divergent series*.

A very important infinite series whose sum can be rigorously determined is called the **harmonic series**. It is the following infinite series:

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$$

10. Write out the first fifteen terms in the harmonic series.

11. Sum the first three terms in the harmonic series.

12. Sum the first seven terms in the harmonic series.

13. Sum the first fifteen terms in the harmonic series.

14. How fast does the sum appear to be growing as you add more terms.

15. Is it clear whether the series will sum to a specific value? If so, what is this value? Consider now the infinite series

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{16} + \dots$$

16. How many terms of $\frac{1}{2}$ are there?

17. What is the sum of these $\frac{1}{2}$ "terms"?

18. How many terms of $\frac{1}{4}$ are there?

19. What is the sum of these $\frac{1}{4}$ "terms"?

20. How many terms of $\frac{1}{8}$ are there?

21. What is the sum of these $\frac{1}{8}$ "terms"?

22. Do the preceding problems suggest how many terms of there will be in this series. How many?

23. What is the sum of these " terms"?

24. If we sum this series through the end of the $\frac{1}{4}$"terms", what will the sum be?

25. If we sum this series through the end of the $\frac{1}{8}$ "terms", what will the sum be?

26. If we sum this series through the end of the " $\frac{1}{16}$ terms", what will the sum be?

27. If we sum this series through the end of the $\frac{1}{32}$ " terms", what will the sum be?

28. The sum of this series is greater than 10. Through what terms would we have to add to reach a sum greater than 10?

29. The sum of this series is greater than 100. Through what terms would we have to add to reach a sum greater than 10?

30. The sum of this series is greater than 1,000,000. Through what terms would we have to add to reach a sum greater than 10?

31. Compare the first term of the harmonic series with the first term of this new series. Then compare the second terms of these series. And the third. And the fourth. Whenever they are not equal, which series has the larger terms?

32. Classroom Discussion: Explain how to use the Investigations above to determine precise sums for both the series on page 1 and the harmonic series.

In addition to its use for series like the Grandi series, mathematicians also call series whose sums approach $\pm\infty$ divergent series.

The proof of the divergence of the harmonic series you just completed is due to Nicole Oresme (French Philosopher, Theologian, Mathematician, and Astronomer; c. 1323 - 1382).

Portrait of Nicole Oresme: Miniature from Oresme's *Traité de l'espere*, Bibliothèque Nationale, Paris, France