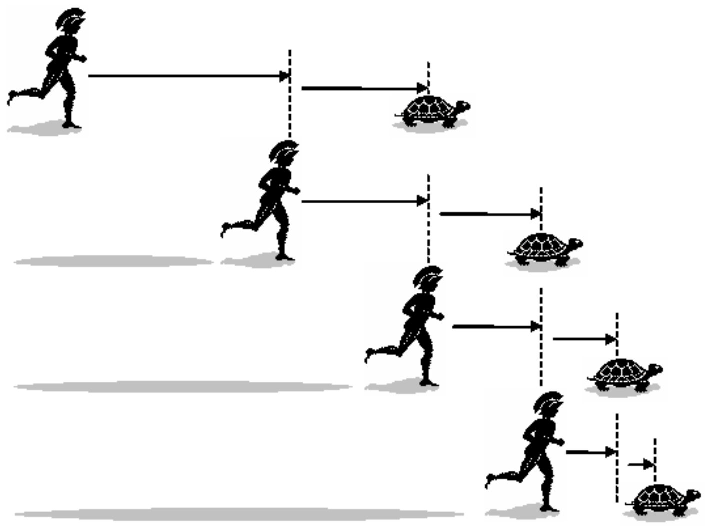


Zeno Revisited

> The fear of infinity is a form of myopia that destroys the possibility of seeing the actual infinite, even though it in its highest form has created and sustains us, and in its secondary transfinite forms occurs all around us and even inhabits our minds.
>
> **Georg Cantor** (German Mathematician;1845 - 1918)

Recently, you developed a formula for the sum of a geometric series. Namely, the sum is given by

$$1 + r + r^2 + r^3 + ... = \frac{1}{1-r}$$

You used this formula to easily determine the sum of four series that previously had been fairly hard to compute. Indeed, the sum of *every* geometric series with $-1 < r < 1$ can be computed easily this way. This is a powerful result.

Next, you will see how to use your results on geometric series to help analyze Zeno's Achilles paradox.

Zeno Redux

As in the earlier activity, assume Zeno's hypothetical racecourse is 100 meters and that Achilles allows the Tortoise a 50 meter lead. When the race started, Achilles needed to reach the 50m point where Tortoise started. When he reached that point, Tortoise had moved further ahead. We can think of this as the first stage of the race. In the second stage Achilles needed to reach Tortoise's new location. When he reached that point Tortoise again had moved further ahead.

These stages continue indefinitely, suggesting that Achilles can never catch Tortoise. This is the paradox.

This is a qualitative analysis. We would like to use what we have learned about series to provide a quantitative analysis. Because this paradox involves both distance and time, speed must be involved. So let us assume that Achilles runs the race at a constant speed of 10 m/sec and Tortoise crawls at a constant speed of 4 m/sec.

1. How long does it take for Achilles to reach the point where Tortoise started the race?

2. Since the beginning gun which started the race, how much time has elapsed?

3. When Achilles reaches the starting point of Tortoise, how much further ahead has Tortoise moved?

4. At this point, how far is Tortoise from the start?

5. How long does it take for Achilles to go from Tortoise's starting location to Tortoise's location at the end of Stage 1 that you found in Investigation 4?

6. Since the beginning gun which started the race, how much time has elapsed?

7. When Achilles reaches the Tortoise's location at the end of Stage 1, how much further ahead has Tortoise moved?

8. At this point, how far is Tortoise from the start?
We need to continue to identify the times and locations involved in the motions of our two racers. One way to do this is with a table.

Stage	Time Interval	Total Elapsed Time	Distance covered by Turtle	Turtle Location	Distance covered by Achilles	Achilles Location
0	0	0	0	50	0	0
1					50	50
2						
3						
4						
5						

6						
7						
8						
9						
10						

9. Use your data from Investigations 2-9 to fill in the rows for the first and second stages.

10. Now determine data which enables you to complete the table for stages 3 - 10. Describe any patterns that you see.

You should notice that both the total elapsed time and the locations of our two runners seem to be *converging* to fixed *limits* as the stage number $n \to \infty$. We would like to determine what these limits are.

11. Using your data, roughly estimate the limits as $n \to \infty$ of the total elapsed time, Tortoise's Location, and Achilles' Location.

12. Write the limit of the Total Elapsed Time as the sum of an infinite series.

13. Use what you have learned about infinite series to determine the sum of this infinite series. Does this limit agree with your estimate above?

14. Write the limit of Tortoise's location as the sum of an infinite series.

15. Use what you have learned about infinite series to determine the sum of this infinite series. Does this limit agree with your estimate above?

16. Check that your previous answer makes sense by using the time in Investigation 12 and the constant rate of speed Tortoise travels.

17. Write the limit of the Achilles's location as the sum of an infinite series.

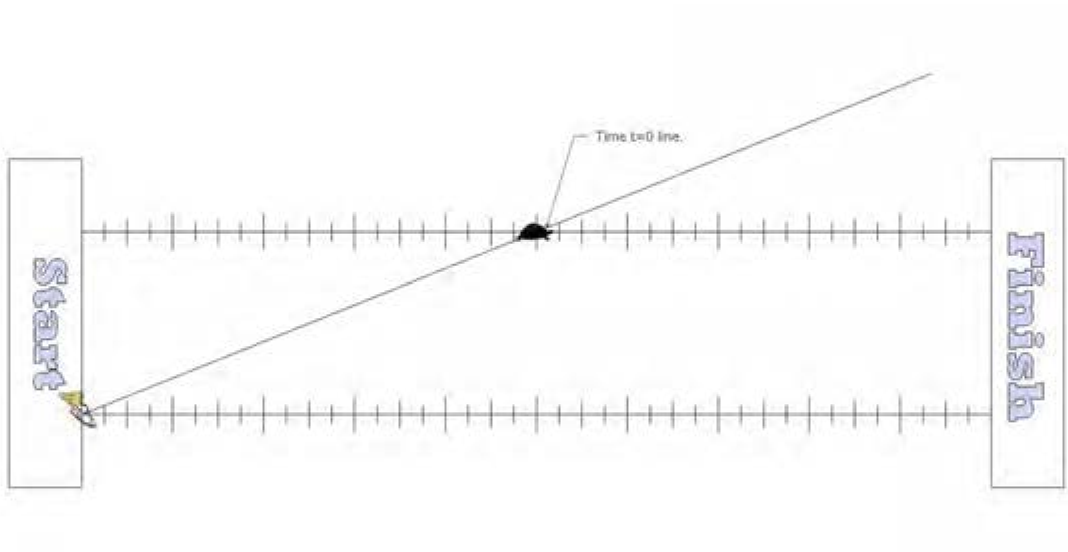
18. Use what you have learned about infinite series to determine the sum of this infinite series. Does this limit agree with your estimate above?

19. Check that your previous answer makes sense by using the time in Investigation 12 and the constant rate of speed Achilles travels.

20. Based on these investigations, what happens at the time in Investigation 12?

21. What happens after the time in Investigation 12?

Another useful way to envision Zeno's Achilles paradox is to consider it graphically.



22. On the image above, plot the location of Achilles and Tortoise at the end of stages 1 - 6. Keep each racer in their own lane. As has been done for time t = 0, draw a line from Achilles' location extending well through Tortoise's location.

23. What do you notice about all of the lines that you have drawn?

24. Your completed figure should now remind you of a perspective drawing. It may also remind you of your analysis of the Wheel of Aristotle. Does this figure shed any additional light on Zeno's Achilles paradox? Explain.

In 1901 Bertrand Russell (English mathematician, logician, and philosopher; 1872 - 1970) remarked:

> Zeno was concerned with three problems...These are the problem of the infinitesimal, the infinite, and continuity...From his to our own day, the finest intellects of each generation in turn attacked these problems, but achieved broadly speaking, nothing...Weierstrass, Dedekind, and Cantor,...have completely solved them. Their solutions...are so clear as to leave no longer the slightest doubt or difficulty. This achievement is probably the greatest of which the age can boast...The problem of the infinitesimal was solved by Weierstrass, the solution of the other two was begun by Dedekind and definitely accomplished by Cantor.

Nonetheless, dissenters remain. In a September, 1994 *Scientific American* feature “Resolving Zeno’s Paradoxes”, William I. McLauglin (American Space Scientist;1935 -) stated:

> At last, using a formulation of calculus that was developed in just the past decade or so, it is possible to resolve Zeno’s paradoxes. The resolution depends on the concept of infinitesimals, known since ancient times but until recently viewed by many thinkers with skepticism.

At last? What does this mean?

26a. Independent Investigation: Interview a mathematicians and report on her/his view of the status of legitimate mathematical solutions to Zeno’s paradoxes.

An issue that further complicates Zeno’s paradoxes is the distinction between idealized, mathematical models of time and space on the one hand and the real time and space we inhabit on the other. Idealized time and space are continuous and infinitely divisible. However, in prevailing physical theories matter is neither continuous not infinitely divisible. That suggests to some that real space and time are not either.

26b. Alternate Independent Investigation: Interview a mathematician, philosopher, and/or physical scientist and report on her/his views on the status of legitimate, real world solutions to Zeno’s paradoxes.