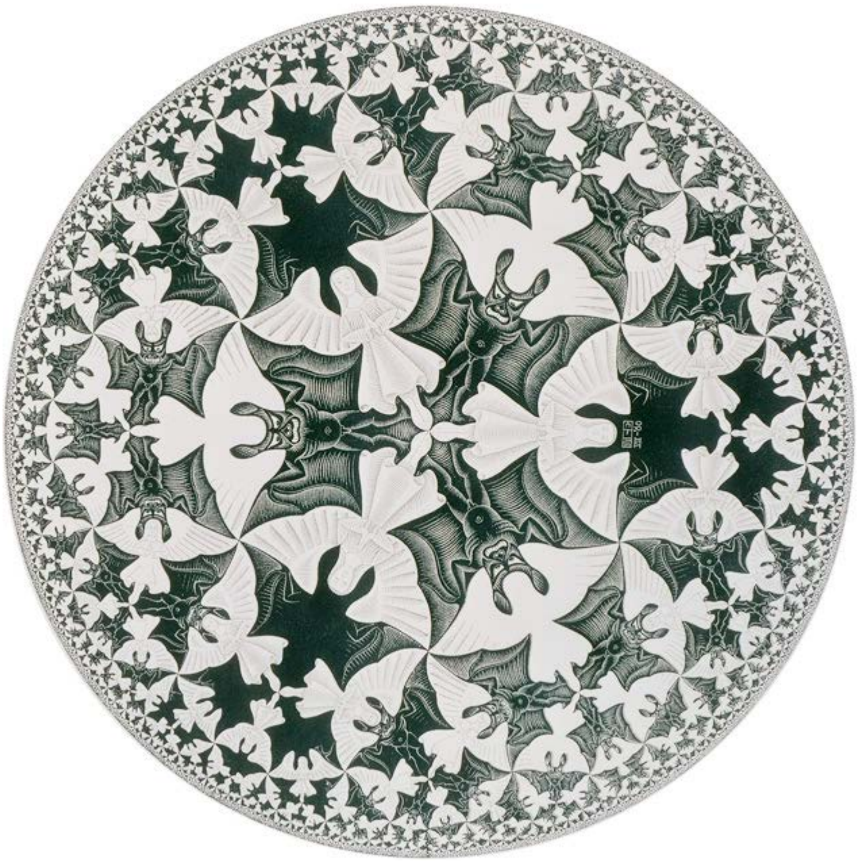


Angels and Demons, 1960
M. C. Escher (Dutch Graphic Artist, 1898 - 1972)

Infinite Geometry

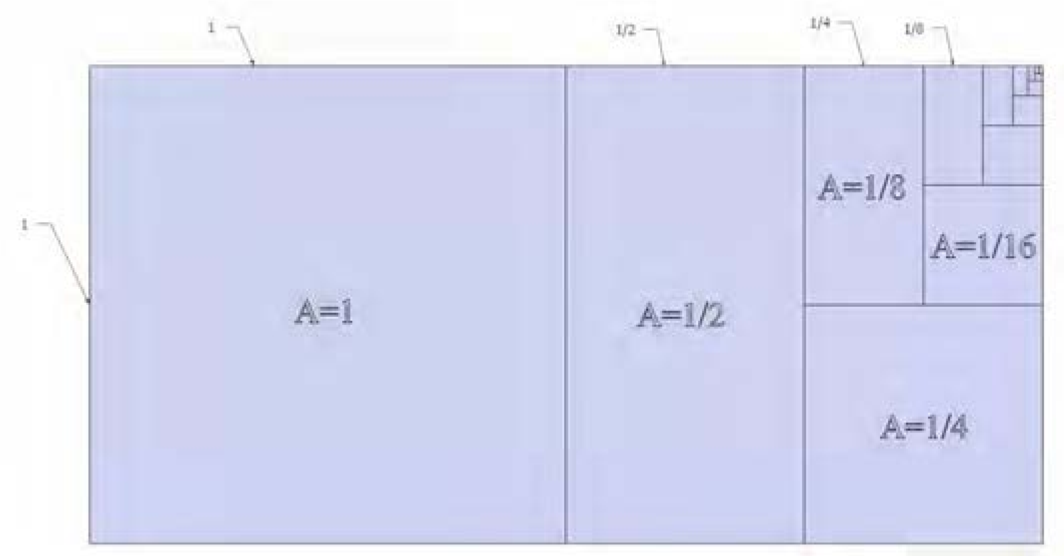
Mathematics is full of "infinite objects" that are not infinite in the sense that they are unlimited or without bound, but only in the sense that they are comprised of infinitely many pieces or created as the result of some infinite process.

M. C. Escher (1898 - 1972) was a Dutch graphic artist who grappled with the challenge of creating a visual representation of the infinite. In the design above, Escher created infinitely many modified copies of interlocking angels and demons that tessellate the plane.[1]

1 For a detailed account of Escher's work with infinity, see "capturing Infinity" by Thomas Witing in the March 2012 edition of Reed Magazine, available online at
http://www.reed.edu/reed_magazine/march2010/features/capturing_infinity/index.html

Summing $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} \ldots$

The figure below shows a large rectangle, running from the left of the image to the right, which has been decomposed into infinitely many squares and rectangles.



1. Several of the rectangles and squares have their areas labelled. What are the dimensions and the area of the large rectangle?

2. What are the dimensions and areas of the two largest rectangles that have not be labelled?

3. What are the dimensions and areas of the two largest squares that have not been labelled?

4. On a copy of the Figure shade in an area equal to $1 + \frac{1}{2}$.

5. On a copy of the Figure shade in an area equal to $1 + \frac{1}{2} + \frac{1}{4}$. Show how this sum, can be geometrically visualized; i.e. how can you see both numerator and denominator using your figure?

6. On a copy of the Figure shade in an area equal to $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}$. Show how this sum, can be geometrically visualized; i.e. how do you see both numerator and denominator using your figure?

7. If you were asked to shade an area equal to $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$, what would you shade?

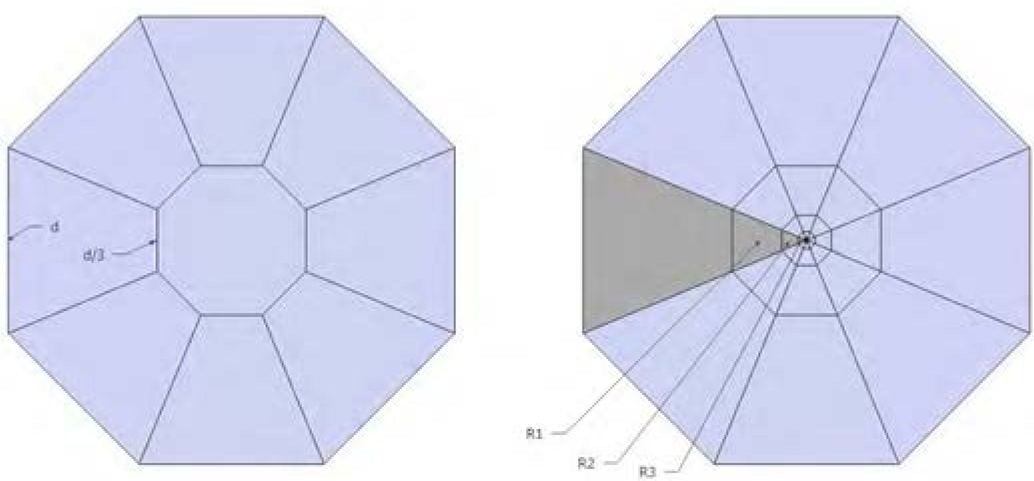
8. Establish the sum of the infinite series $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} \ldots$ geometrically.

Your explorations to determine the sum of the infinite series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$ make up what is generally called a **proof without words**. There are many beautiful proofs without words that help establish the sum of infinite series. The investigations below consider one that is a bit harder, but is particularly nice because it works not just for one infinite series but infinitely many.

Geometric Scaling

The left image below[2] shows a regular octagon which has been decomposed into trapezoids and a scaled octagon in the center. The area of the large octagon is 1 square unit and the length of the sides is represented by d. (It turns out that $d = \frac{1}{\sqrt{2\,(1+\sqrt{2})}}$, but this is unimportant.)

The
smaller center-octagon has been scaled down by a factor of $\frac{1}{3}$.



Scaling has been an important tool in our analysis of the infinite. It is here too. We need to discover what happens to areas when the objects that have been measured are scaled up or down.[3]

[2] From "Proof without words: Geometric series formula" by James Tanton in *College Mathematics Journal*, vol. 39, #2, March, 2008, p. 106.
[3] What we will find is a special case of a more general principle that is a fundamental result from geometry. The relationship between scaling and lengths, areas, or volumes gives rise to *fractals*.

9. Draw several planar geometric figures whose areas you can readily find. Label their dimensions and find their areas.

10. Scale down each of the regions in Investigation 9 by a factor of $r = \frac{1}{3}$. Draw the resulting figures and label their new dimensions.

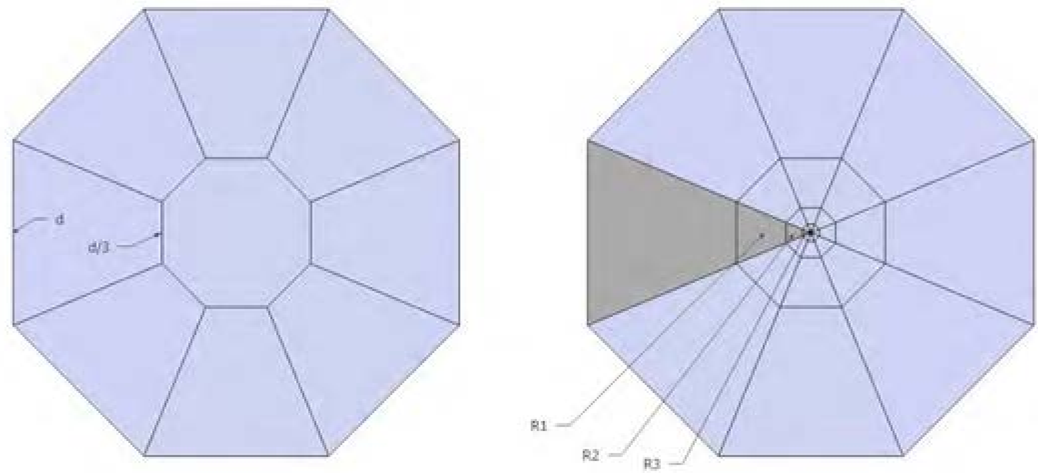
11. Find the area of each of the scaled regions in Investigation 10. How are the areas of the scaled regions related to the areas of the original, unscaled regions?

12. Now scale up the regions in Investigation 9 by a factor of r = 2. Draw the resulting figures and label their new dimensions.

13. Find the area of each of the scaled regions in Investigation 12. How are the areas of the scaled regions related to the areas of the original, unscaled regions?

14. There is a general relationship between the areas of planar regions and the areas of these regions once they have been scaled. Use your examples above to complete the statement of this relationship:

    Theorem 1. Let R be a region in the plane whose area is A. If R is scaled by a factor of m then the area of the scaled region is _________ × A.



Summing $\frac{1}{9}+\frac{1}{9^2}+\frac{1}{9^3}+\frac{1}{9^4}+\ldots$

15. What is the area of the octagon at the center of the figure on the left in the Figure above?

16. Determine the area of each of the quadrilaterals in the figure on the left in Figure above.

The octagon on the right in the Figure above is congruent to the octagon on the left. It has been decomposed indefinitely, the lengths of the sides of the nested octagons being $d, \frac{d}{3}, \frac{d}{9}, \frac{d}{27}, \ldots$

17. Determine the area of the region which is labelled R_2.

18. Determine the area of the region which is labelled R_3.

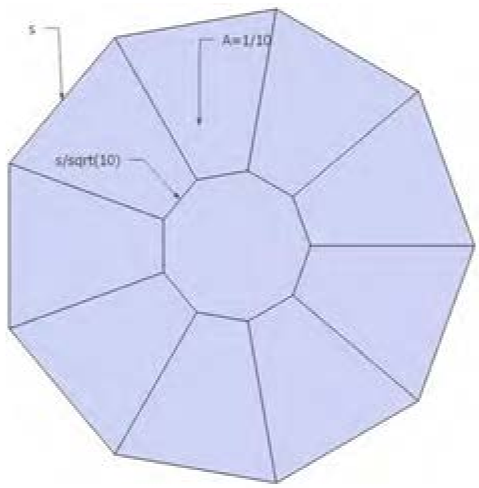
19. Determine the area of the region which is labelled R_4.

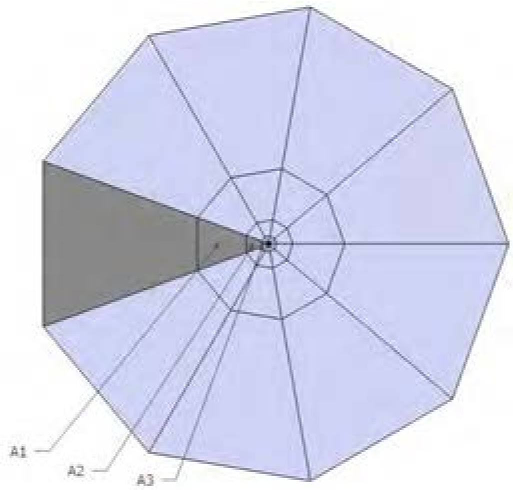
20. You should see a pattern in Investigations 17-19. Will this pattern continue indefinitely? Explain how you know this.

21. On the right figure in Figure 3.2 a single triangle has been shaded. How does the area of this triangle compare to the area of the entire octagon? Explain.

22. Explain how Investigations 15-20 allow you to determine the sum of the infinite series $\frac{1}{9}+\frac{1}{9^2}+\frac{1}{9^3}+\frac{1}{9^4}+\ldots$ as a single fraction.

0.999... Again





23. Using the shapes in Figure above and your method in Investigations Investigations 15-22, determine the sum of the infinite series $\frac{1}{10}+\frac{1}{10^2}+\frac{1}{10^3}+\frac{1}{10^4}+\ldots$ as a single fraction.

24. Convert each of the individual terms in the series $\frac{1}{10}+\frac{1}{10^2}+\frac{1}{10^3}+\frac{1}{10^4}+\ldots$ to decimals.

25. How hard is it to determine the sum of the infinite series $\frac{1}{10}+\frac{1}{10^2}+\frac{1}{10^3}+\frac{1}{10^4}+\ldots$ as a decimal?

26. You now have determined the value of the sum of the infinite series as a fraction and as a decimal. Multiply both by 9. What does this tell you about the identity of 0.999...?

> It is a curious thing that people on the whole do not boggle over an infinite decimal of this kind (1.111111111...), but when they look at an infinite addition like this:
>
> 1 + 1/10 + 1/100 + 1/1000 + 1/10000 + ...*ad infinitum*,
>
> although this is just another way of writing the other. But I am not surprised at their looking aghast at the latter, though rather surprised that they accept the former[4]
>
> **Rosza Peter** (Hungarian Mathematician;1905 - 1977)

27. Are you “boggled” or “aghast” about 0.999...? Describe which approach is most compelling or what is still lacking in your ability to understand 0.999....

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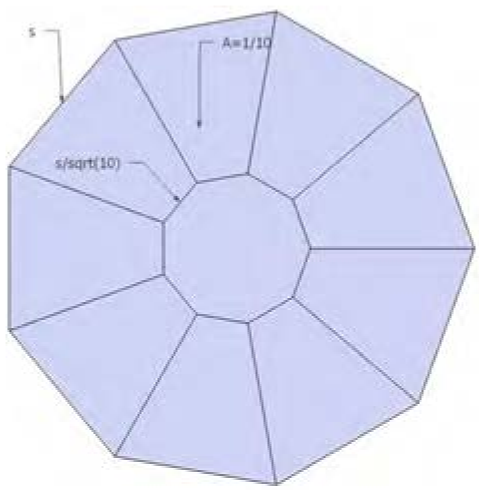
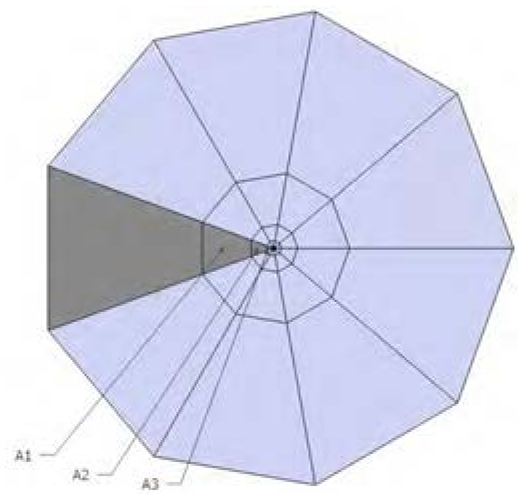
[4] From Playing with Infinity: Mathematical Explorations and Excursions, p. 104.





The Sum of a Geometric Series

You will conclude the Investigations by generalizing the results on infinite series from the previous sections.

28. In Investigation 22 and Investigation 23 you determined the sums of the infinite series $\frac{1}{9}+\frac{1}{9^2}+\frac{1}{9^3}+\frac{1}{9^4}+\ldots$ and $\frac{1}{10}+\frac{1}{10^2}+\frac{1}{10^3}+\frac{1}{10^4}+\ldots$. Do you think that this geometric method can be used to determine the sum $\frac{1}{5}+\frac{1}{5^2}+\frac{1}{5^3}+\frac{1}{5^4}+\ldots$? If so, what do you think the sum will be?

29. What about $\frac{1}{8}+\frac{1}{8^2}+\frac{1}{8^3}+\frac{1}{8^4}+\ldots$?

30. What about $\frac{1}{37}+\frac{1}{37^2}+\frac{1}{37^3}+\frac{1}{37^4}+\ldots$?

31. We hope that you see a pattern. Complete the following:

> **Theorem 2.** For any positive whole number the infinite series $\frac{1}{k}+\frac{1}{k^2}+\frac{1}{k^3}+\frac{1}{k^4}+\ldots$ *converges*.
> Moreover, its sum is given by $\frac{1}{k}+\frac{1}{k^2}+\frac{1}{k^3}+\frac{1}{k^4}+\ldots=$____________.

32. Explain how you can "see" this result geometrically.

33. Below it will be useful if our infinite series began with the term 1+. Adapt the theorem just stated to find a sum of these series - simplifying the sum as much as you can:

Theorem 3. For any positive whole number the infinite series $1+\frac{1}{k}+\frac{1}{k^2}+\frac{1}{k^3}+\frac{1}{k^4}+\ldots$ converges.

Moreover, its sum is given by $1+\frac{1}{k}+\frac{1}{k^2}+\frac{1}{k^3}+\frac{1}{k^4}+\ldots=$___________.

Let r denote some arbitrary number. The infinite series $1+ r + r^2 + r^3 +\ldots$ is called a **geometric series**.

34. Explain why each of the infinite series you have considered above are geometric series by finding an appropriate value of r for each.

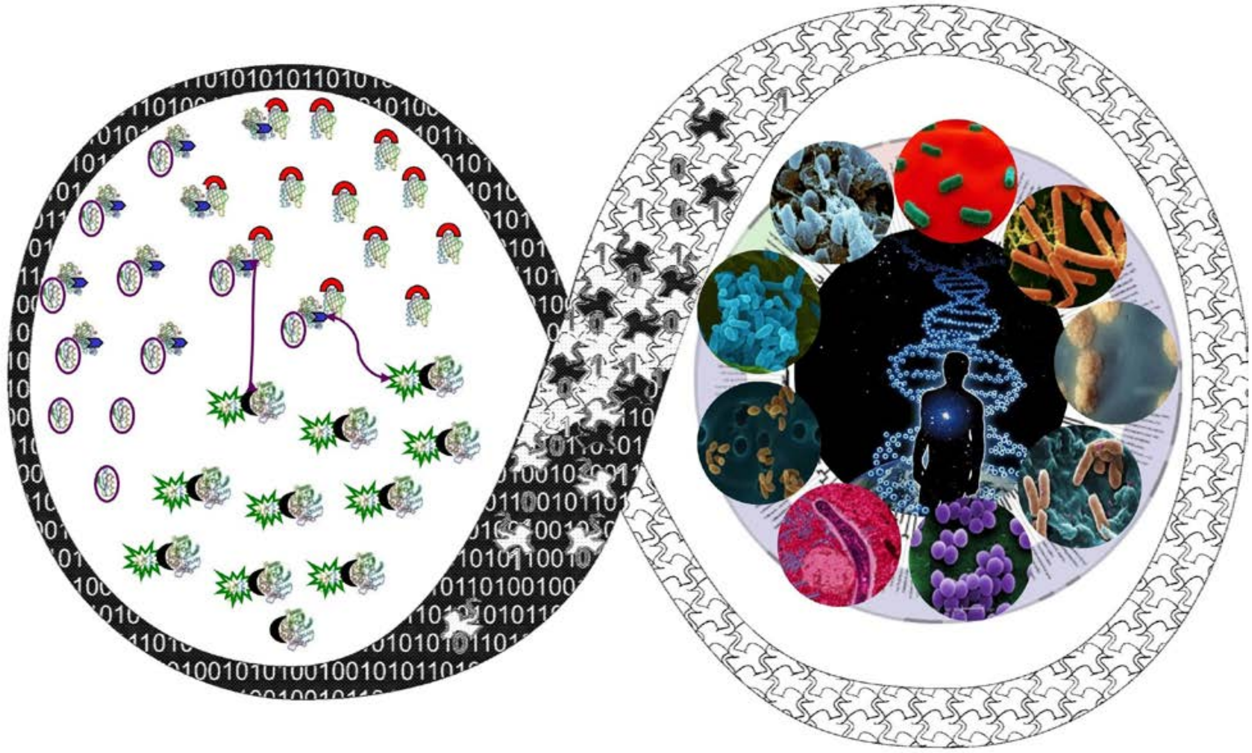
35. How are consecutive terms in a geometric series related to one another?

36. What do you notice about all of the values of r in Investigation 34?

37. For certain values of r it should be clear that the sum of the associated geometric series is infinite. For example, $1 + 1 + 1^2 + 1^3 + \ldots = 1 + 1 + 1 + 1 + \ldots = \infty$. Find several other values of r for which the geometric series clearly *diverges* to ∞?

38. In contrast, for what values of r does it seem possible that the sum of the geometric series is finite? Explain.

39. Classroom Discussion: Compare results in Investigation 37 and Investigation 38. Together can you form a conjecture that describes precisely what values of r will result in a geometric series which diverges and what values of r will result in a geometric series that converges to a finite sum?



Infinite Image 1.0
Ram Samudrala (American Scientist, Musician Philosopher, 1972 -)
http://www.ram.org