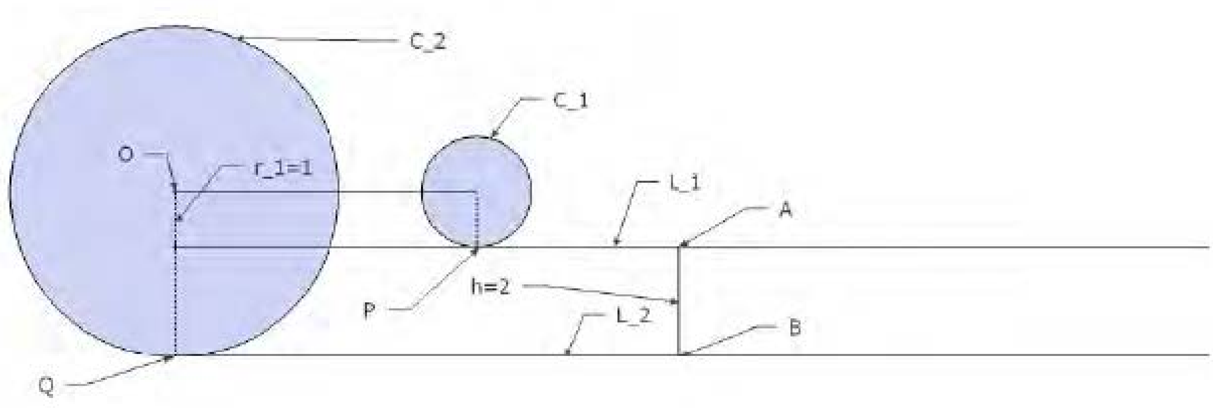
Infinite Paradox

A **paradox** is a statement or result that seems to be logically well founded although it is contrary to common sense or intuition. Aristotle's wheel paradox is from the Greek work *Mechanica* traditionally attributed to Aristotle (Greek philosopher and scientist, 384 - 322 BCE). There are two wheels, one within the other, whose rims take the shape of two circles with different diameters. The wheels roll without slipping for a full revolution. The paths traced by the bottoms of the wheels are straight lines, which are apparently the wheels' circumferences. But the two lines have the same length, so the wheels must have the same circumference, contradicting the assumption that they have different sizes: a paradox.

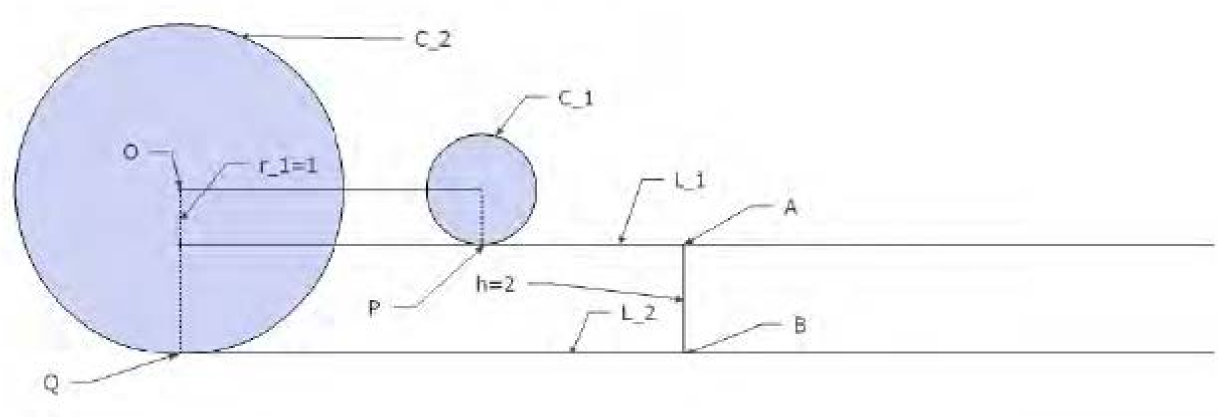
Intrigued? Let's investigate further!

The Wheel of Aristotle

Consider the two disks pictured in Figure 2.3 whose boundaries are the circles C_1 and C_2 and whose radii are $r_1 = 1$ unit and $r_2 = 3$ units respectively. Throughout, think of these as actual physical wheels which are rolling along the parallel lines L_1 and L_2 respectively. The points P and Q travel along the circles as they roll.



Two Rolling Wheels



1. For the first part of Investigation 1, allow the smaller wheel to roll to the right through one full rotation.

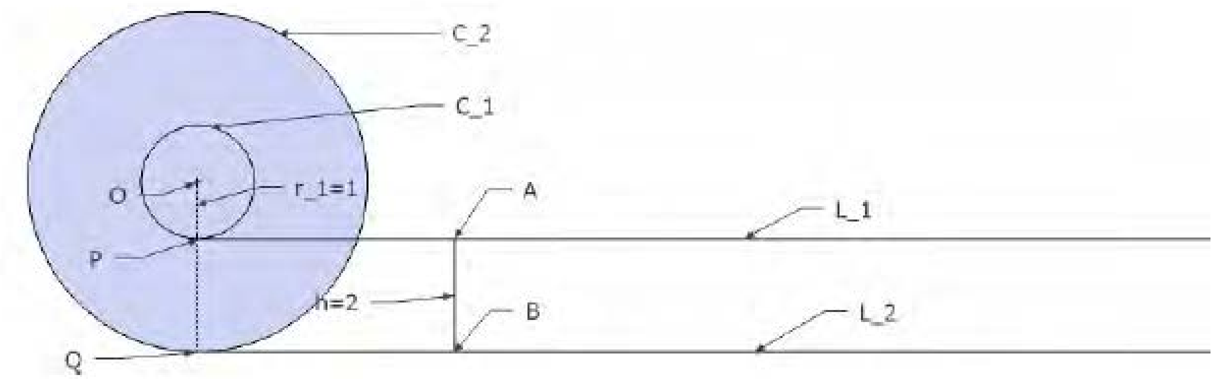
    a. Draw a scale picture of the resulting situation, labelling the point where P first comes in contact again with L_1 by C.

    b. How long is the line segment $\underline{AC}$? Calculate using the Circumference formula. Show your work.

    c. For the second part of Investigation 1, allow the larger wheel to roll to the right through one full rotation. Draw a scale picture of the resulting situation, labeling the point where Q first comes in contact again with L_2 by D.

    d. How long is the line segment $\underline{BD}$? Calculate using the Circumference formula. Show your work.

The actual Wheel of Aristotle is constructed by gluing two circular disks together with their centers concurrent[1]. The vertical line segment $\underline{OQ}$ is simply a reference line from the center of the circles to L_2.



2. Investigation 2: Consider the paths followed by each circle when they are joined in a Wheel of Aristotle, pictured above in its starting position.

Now, analyze the situation as the Wheel of Aristotle "rolls" along the lines L_1 and L_2.

a. Allow the Wheel of Aristotle to roll horizontally to the right until the first time that the line $\underline{OQ}$ is again vertical and is in contact with L_2. Draw a scale picture of the resulting situation, labeling the point where *P* comes into contact with L_1 by *C* and the point where *Q* again comes in contact with L_2 by *D*.

b. When you rolled the Wheel of Aristotle in Investigation 2a, how many rotations did the larger wheel undergo? Explain.

c. When you rolled the Wheel of Aristotle in Investigation 2, how many rotations did the smaller wheel undergo? Explain.

d. What is the length of the line segment $\underline{AC}$?

e. What is the length of the line segment $\underline{BD}$?

f. Compare your answers to Investigation 1. Describe the disagreement you see. This difficulty is often referred to as the *Wheel Paradox of Aristotle*.

g. Can you explain how and why the disagreement arises?

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[1] Two points are **concurrent** if they occupy the same point in space.

Investigation 3

The apparent contradiction that arose in Investigation 1 and 2 can be explained physically. Construct a *Square Wheel of Aristotle* from paper or cardboard that mimics the construction above but uses squares instead of circles. We will keep the notation L_1 and L_2 from above.

1. Draw a time-lapse like picture which shows how the large square travels as the Square Wheel of Aristotle undergoes one full rotation. When is the large square in contact with L_2? How does the size of this contact area related to the size of the square?

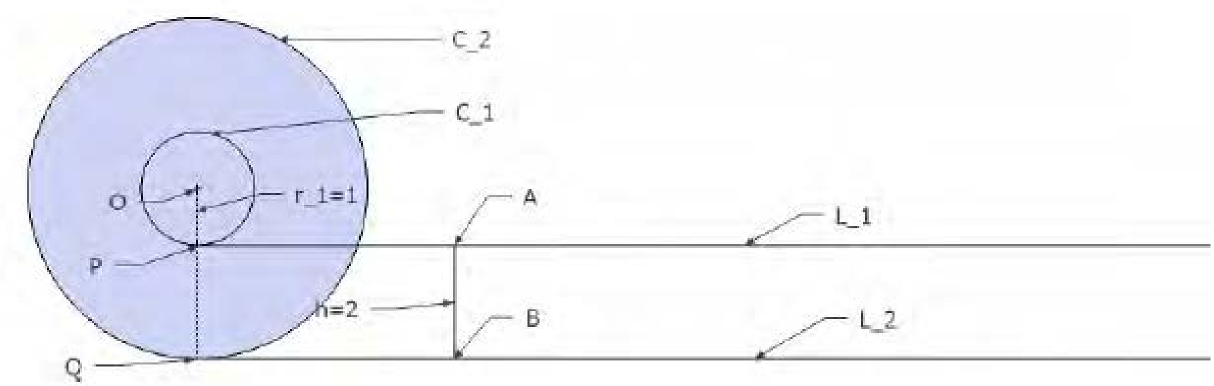
2. Similarly, draw a time-lapse like picture which shows how the small square travels as the Square Wheel of Aristotle undergoes one full rotation. When is the small square in contact with L_1? How does the size of this contact area related to the size of the square?

3. Is the behavior of the Square Wheel of Aristotle more understandable than the (circular) Wheel of Aristotle? Explain.

4. Could the construction and analysis carried out above be carried out if one employed wheels constructed from hexagons, octagons, dodecagons, etc.? If so, what would the results be? Describe them carefully.

5. What explanation for the apparent contradiction considered in Investigation 1 and 2 do these examples suggest? Is this a legitimate answer that explains the apparent contradiction? Explain.

There is another strange situation that the Wheel of Aristotle brings to light - one that will become essential to our study of the infinite.



6. How many points make up the circle C_1?

7. How many points make up the circle C_2?

8. Which do you think contains more points circle C_2 or circle C_1? How many more? Suppose the line segment OQ is free to rotate about the point O, like the hand of a clock.

9. As the line segment OQ rotates, how many points on C_1 is it in contact with at any given time? How many points on C_2 is it in contact with at any given time?

10. Can each point on C_2 be “matched up” with a unique point on C_1 by suitably rotating the line segment OQ? Explain.

11. Can each point on C_1 be “matched up” with a unique point on C_2 by suitably rotating the line segment OQ? Explain.

12. What do Investigations 9-11 tell you about the relative number of points on C_2 as compared to the number of points on C_1?

13. How do your answers to Investigation 8 and Investigation 12 compare? Explain in detail.