...and Beyond

Purely mathematical inquiry in itself...lifts the human mind into closer proximity with the divine than is attainable through any other medium. Mathematics is the science of the infinite, its goal the symbolic comprehension of the infinite with human, that is finite, means.
Hermann Weyl (German Mathematician; 1885 - 1955)

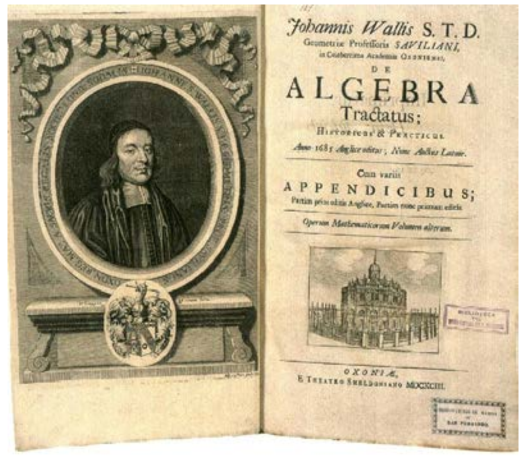
In the last activity you explored very large numbers. You saw that the number of molecules in the observable universe is less than 10^{107}. You also saw how gigantic this number was in human terms. Yet this number was simply a garden-variety large number that was quickly eclipsed by many of the truly large numbers. Many of these numbers are well beyond human comprehension in a meaningful physical way. Still, no matter how much they stretch the limitations of comprehension, all those numbers are **finite**.

Do we have any real way of understanding the **infinite**? If you think not, you would find yourself in good company among philosophers, mathematicians and scientists through the nineteenth century. However, things change rapidly after that, as we shall see.

Let us begin our study of the infinite in a very familiar setting, with the natural numbers: 1,2,3,.... The infinitude of this collection of numbers is natural and obvious. Even the smallest children, once they understand the central idea of counting, demonstrate their understanding of the infinitude of this set by gloating that they can always find a number bigger than any you might pick. “Plus one,” they reply smugly.

The most commonly used symbol for the infinite is:

∞



This symbol was first introduced by John Wallis (English mathematician; 1616 - 1703) in his important *Tractatus de sectionibux conicis* and *Arithmetica infinitorum*, both appearing in 1655.

Why did he choose this symbol?
Is this an appropriate symbol?

Investigation 1

1. The most commonly used Roman numeral for 1,000 is **M**. However, another prominent symbol was **ↀ**. Explain how the Roman symbol **ↀ** may have given rise to the symbol ∞.

2. The most commonly used Roman numeral for 500 is **D**. How might this symbol be related to **ↀ** ?

3. To make numbers larger than 1,000 additional circles were added. So **ↀↀ** was 10,000. What would 100,000 be? How long could this process be continued?

Karl Menninger (German mathematics teacher; 1898 - 1963) tells us in his encyclopedic Number Words and Number Symbols: A Cultural History of Numbers that “at one time 100,000 was the ‘last’ number known to the Romans.”

4. How does Wallis’ choice of a ∞ for a symbol for the infinite relate to this historical fact?

5. Throughout history numerals used to represent numbers have often been closely related to the words used to represent the number. For example, the Roman numeral **C** comes from the Latin *centum* for hundred.

   a) What’s the last (lowercase) character of the Greek alphabet? How is this symbol visually related to ∞?

b) Might this have something to do with the genesis of the use of the symbol ∞? Explain.

6. In the days of Wallis, movable type pieces - small blocks of metal with a raised letter, numeral, or figure on the top - were used in all printing presses. How might the use of the symbol ∞ be related to the extant type pieces of the time?

The shape of the symbol ∞ is now known as a **lemniscate**. This term comes from the Latin *lemniscus*, which means hanging ribbon. Interestingly, this term was coined, in a mathematical sense, to name an algebraic curve discovered by Jakob Bernoulli (Swiss mathematician; 1654 1705) in 1694 - almost 40 years after Wallis had begun using it as a symbol for the infinite. While the lemniscate is more widely known as the symbol for infinity, it has a much more important role in mathematics - part of the family of curves which helped give rise to the study of *elliptic functions*; profoundly important mathematical objects. Additionally, the lemniscate is a special type of Cassinian oval a family of curves which were discovered by Giovanni Domenico Cassini (Italian mathematician and engineer; 1625 - 1712) in 1680. So the history of this symbol is somewhat scattered.

7. What do you think of the symbol ∞ for the infinite? Do you like it? Do you find it aesthetically pleasing? Does it seem to be an appropriate symbol? If not, can you think of an alternative? Explain.

8. So how does the symbol represent a “hanging ribbon”? From a sheet of paper cut two strips each about ½ inch wide and about 12 inches long. Bring the two ends together to form a short cylinder - a loop.

9. Tape one loop together. Take the other loop and give one end a half twist. Now tape the ends together. How do these figures differ from each other? Explain.

10. You’ve created what is known as a Möbius strip[1]. Pick it up and hold one part of the ribbon between your thumb and forefinger. Turn it around, viewing it from several different vantage points. Can you see the infinity symbol represented?

11. The Möbius strip is actually a marvelous and surprising mathematical object. How many “sides” do you think the Möbius strip has? With a pen or a pencil, draw a stripe down the middle of the Möbius strip. Continue drawing, without picking your pen or pencil up, until

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[1] The Möbius strip was invented by August Ferdinand Möbius (German mathematician and astronomer; 1790 - 1868). M¨obius, and those that attached his name to this object, were unaware that had been indepedently invented, and actually described in print earlier, by Johann Benedict Listing (Czech mathematician; 1808 1882).

you have returned to your starting point. What do you notice?

12. Lay your Möbius strip back down. By folding it in three places, can you make it into a triangle?
13. The universal recycling symbol shown below. Your folded Möbius strip should resemble it. Is the symbol actually identical to your band? Explain precisely.

Sets and Infinity

So what kinds of things are actually infinite? The infinite objects we will consider here will generally be **sets** which are made up of infinitely many **elements**. So what is a set? And what are elements?

Georg Cantor (German Mathematician; 1845 - 1918), the *Father of the Infinite*, said we should think of a set intuitively as "a Many that allows itself to be thought of as a One." For example, the natural numbers 1,2,3,... form the infinite set $N = \{1,2,3,...\}$. We think of the natural numbers as one object, but of course this one object is a collection of infinitely many other objects. As you work with sets here we urge you to carefully distinguish between sets and the objects that are collected together to form these sets. The objects that are collected together to form the set are called its **elements**. You can think of the braces that contain them in the standard notation as hands that are collecting the set's elements together, holding them as a one.

To describe a set we will often list its elements. But we certainly cannot list all of the elements of an infinite set. So we will often list typical elements and then will then use the **ellipsis** "..." to denote the continued extension of these elements when the nature of their continuation should be clear.

A Set of Set Exercises

Like numbers, *functions*, and most other mathematical objects, there are many ways to perform operations on sets. One can compare them (with *inclusion* denoted by ⊂ and ⊆), combine them (with *union* denoted by □), and find common elements (with *intersection* denoted by ∩). Here we will concern ourselves with the **set difference**, denoted by *A* − *B*, where we remove elements of the set *B* from the set *A*. Specifically,

A − *B* = {All elements of A that are not elements of B}.

14. Determine the set **{♦,♥,♣,♠} − {♣,♠}**.

15. Determine the set {●, ◆, ⬬, ▮, ■, ▲} − {Four sided shapes}.

16. Write out ten elements in the set {a,b,c,...,x,y,z} that have been left out by the ellipsis.

17. A good name for the set {a,b,c,...,x,y,z} is EnglishAlphabet. Determine the set EnglishAlphabet − {a, e, i, o, u}?

   a) What is a good name for the set we have removed?

   b) What is a good name for the resulting set that is left?

For finite sets the **cardinality** of the set is the number of elements in the set. For example, the cardinality of the set EnglishAlphabet is 26.

18. Determine the cardinality of each of the sets in Investigation 14, Investigation 15, and Investigation 17.

Each time we use set difference, it gives rise to an *arithmetical statement*. For example, DaysoftheWeek − Weekend = {Monday, Tuesday, Wednesday, Thursday, Friday} gives rise to the arithmetical statement: 7 − 2 = 5.

19. Determine the arithmetical statement that corresponds to the set difference in Investigation 14.

20. Determine the arithmetical statement that corresponds to the set difference in Investigation 15.

21. Determine the arithmetical statement that corresponds to the set difference in Investigation 17.

Infinite Arithmetic

Now let us move to the infinite. In particular, we are interested whether there is an arithmetic of the infinite that is consistent and makes sense.

22. Can you think of an appropriate value for $\infty - \infty$ is? Explain.

23. For each of the sets {1,2,3,...} and {4,5,6,...} write out a dozen elements that have been left out by the ellipses.

24. From the infinite set {1,2,3,...} remove the infinite set {4,5,6,...}. What is the remaining set and how large is it?

25. What does Investigation 24 suggest as a value for $\infty - \infty$?

26. Write out a dozen elements that have been left out of the set {5,6,7,...} by the ellipsis.

27. From the infinite set {1,2,3,...} remove the infinite set {5,6,7,...}. What is the remaining set and how large is it?

28. What does Investigation 27 suggest as a value for $\infty - \infty$?

29. Can the investigations above be extended to suggest other values for $\infty - \infty$? Explain in detail what values are suggested and/or what the limitations are.

30. Write out a dozen elements that have been left out of the set {2,4,6,...} by the ellipsis.

31. From the infinite set {1,2,3,...} remove the infinite set {2,4,6,...}. What is the remaining set and how large is it?

32. What does Investigation 31 suggest as a value for $\infty - \infty$?

33. What do the preceeding investigations suggest about our ability to discover a straightforward arithmetic for ∞?