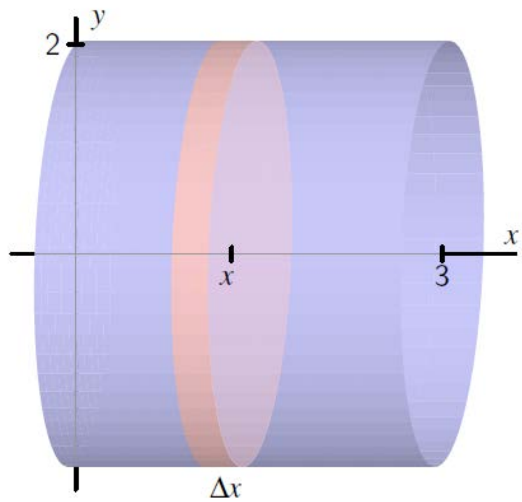
3.5e Integral Volume



Using Definite Integrals to Find Volume

Just as use definite integrals to add the areas of rectangular slices to find the exact area that lies between two curves, you can use integrals to determine the volume of certain regions that have cross-sections of a consistent shape.

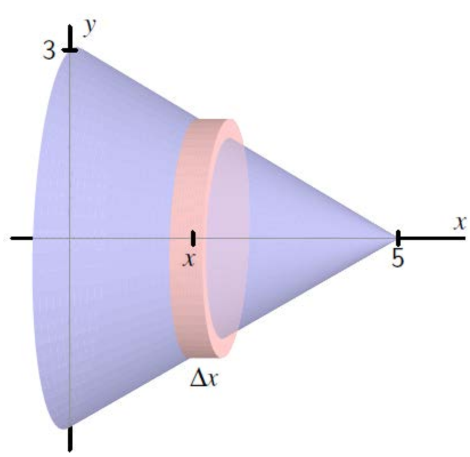
Consider a cylinder of radius 2 and height 3, as pictured above. The formula for the volume of a cylinder is $V = \pi r^2 h$. If you slice the cylinder into thin pieces, you see that each piece is a cylinder of radius $r = 2$ and height (thickness) $\triangle x$. Hence, the volume of a representative slice is

$$V_{slice} = \pi \cdot r^2 \cdot \Delta x \,.$$

As the disks become infinitely thin (as $\Delta x \to 0$) and using a definite integral to add the volumes of the slices, you find that

$$V = \int_0^3 \pi \cdot 2^2 \, dx$$

Moreover, since $\int_0^3 4\pi \, dx = 12\pi$ units^3, you have found that the volume of the cylinder is 12π u^3.



Investigation 1: Consider a circular cone of radius 3 and height 5, which we view horizontally as pictured at right. Use a definite integral to determine the volume of the cone.

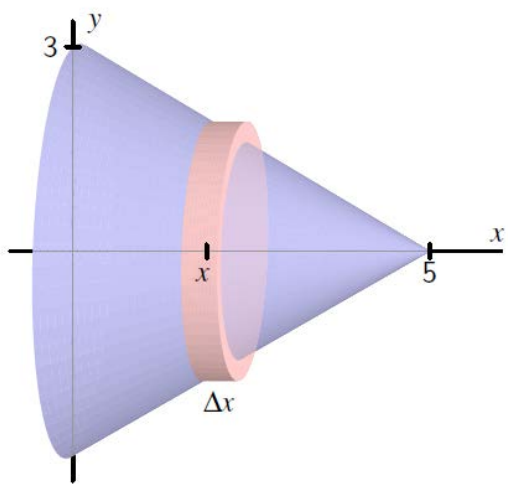
a) Find a formula for the linear function $y = f(x)$ that is pictured at right.

b) For the representative slice of thickness x that is located horizontally at a location x (somewhere between $x = 0$ and $x = 5$), what is the radius of the representative slice? Note that the radius depends on the value of x.

c) What is the volume of the representative slice you found in (b)?

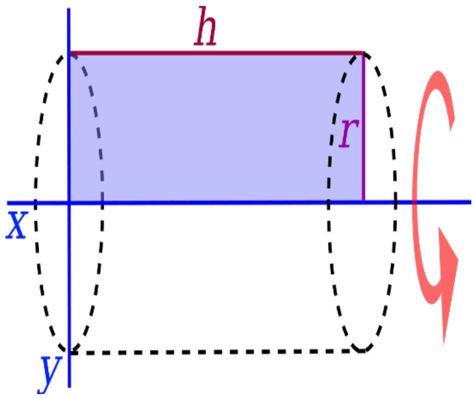
d) What definite integral will sum the volumes of the thin slices across the full horizontal span of the cone? What is the exact value of this definite integral?

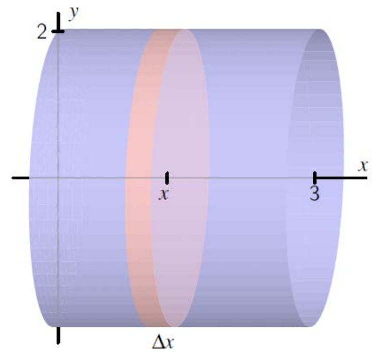
(e) Compare the result of your work in (d) to the volume of the cone that comes from using the formula $V_{cone} = \frac{1}{3}\pi r^2 h$.



II. The Volume of a Solid of Revolution

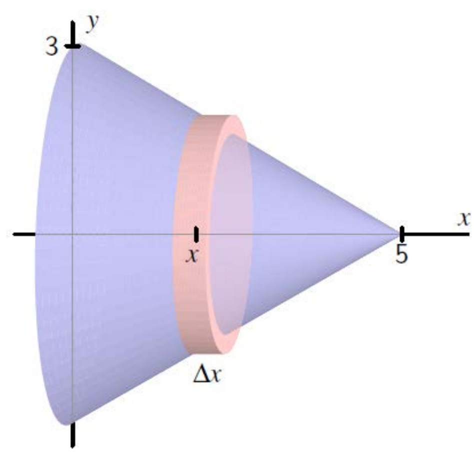
A solid of revolution is a three-dimensional shape that can be generated by revolving one or more curves around a fixed axis. For example, imagine the circular cylinder below right as a solid of revolution: as illustrated below left this 3-d cylinder is generated by revolving the shaded region below the line segment h from 0 to 3.





Note that the cylinder on the right would result from the Area beneath $y=2$, on the interval [0,3].

Investigation 2: Imagine the cone on the right as a solid of revolution.



2a) Sketch a rough model of the rotation, similar to the cylinder model above left.

b) What is the equation of the line that generates the solid.

c) What interval was used?

d) What would the shape look like on the interval [0,4]?

e) Imagine the solid of revolution generated by rotating the area under $f(x) = 2\sqrt{x}$ around the x-axis on the interval $4 \leq x \leq 10$?

i) Describe the object in words.

ii) Sketch the object below. (Or sketch a model of revolution.)

iii) Open the GeoGebra app https://ggbm.at/kQvfYzta .

a) Slide the *Slide Me* control to see the revolution of the given interval

b) Adjust the slider labeled $n = 3$. Describe, in detail, what effect on the solid this makes.

c) Try entering a different equation and exploring the app. What shape is formed by rotating $f(x) = .05x^2$ around the x-axis on the interval $0 \leq x \leq 15$?

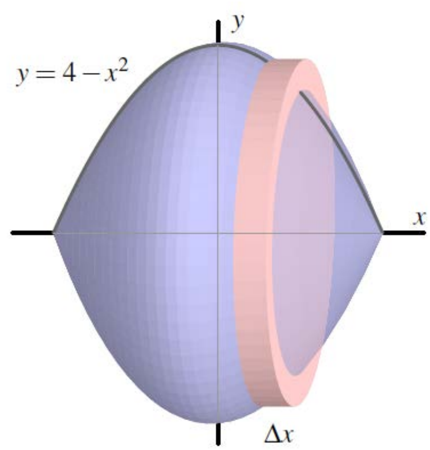
d) Create your own shape. Record the details below and insert a screenshot.

Function:

Interval:

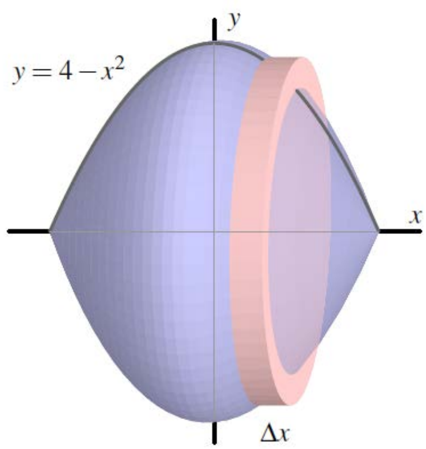
So, what about the volume? There are two primary methods of calculating the area of a solid of revolution. The first is called the ***disk method***, because the reference "slice" is a cylinder with a thickness of $\triangle$x and a circular face, whose radius can be determined.

Example 1. Find the volume of the solid of revolution generated when the region R bounded by y = 4 - x^2 and the *x*-axis is revolved about the *x*-axis.



Solution.

First, we observe that y = 4 x^2 intersects the x-axis at the points (2, 0) and (2, 0). When we take the region R that lies between the curve and the x-axis on this interval and revolve it about the x-axis, we get the three-dimensional solid pictured at right.



Taking a representative slice of the solid located at a value x that lies between $x = 2$ and $x = 2$, you see that the thickness of such a slice is x (which is also the height of the cylinder-shaped slice), and that the radius of the slice is determined by the curve $y = 4 - x^2$.

So, use the Volume formula of a cylinder ($V = \pi r^2 h$) and replace r with ($4 - x^2$).

$$V_{slice} = \pi(4 - x^2)^2 \Delta x,$$

Using a definite integral to sum the volumes of the representative slices, it follows that

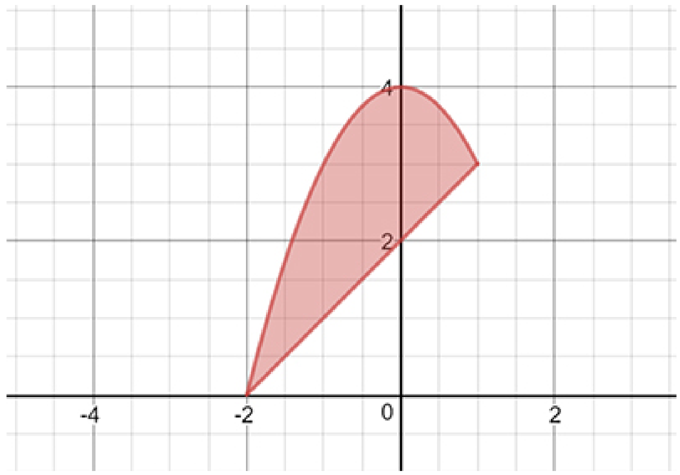
$$V = \int_{-2}^{2} \pi(4 - x^2)^2 \, dx$$

It is straightforward to evaluate the integral and find that the volume is V = 512 units^3.

The general principle of the disk method follows.

> If $y = r(x)$ is a nonnegative function on [a,b], then the volume of the solid of revolution generated by revolving the curve about the x-axis over this interval is given by
>
> $$V = \int_{a}^{b} \pi[r(x)]^2 \, dx$$

A different type of solid can emerge when two curves are involved, as seen in the next example.



Example 2. Find the volume of the solid of region R that lies between $y = 4 - x^2$ and $y = x + 2$ is revolved about the x-axis.

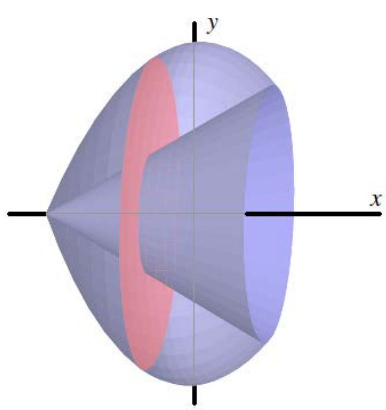
While it is not necessary to graph region R, it is helpful here: can you imagine how the shaded region above would generate the solid below? The solid looks like a Taj Majal-like shape with a cone cut out. The slices will also have a piece cut out, resembling washers.

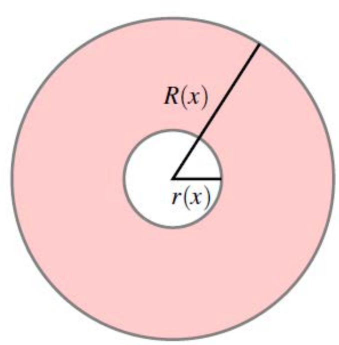
Solution.

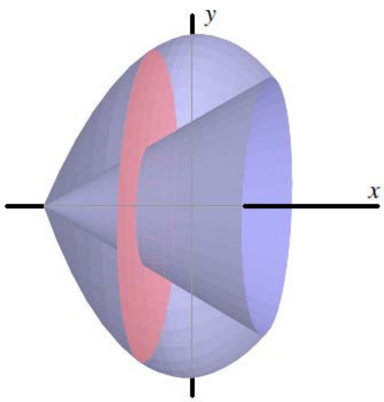
First, determine where the curves $y = 4\ x2$ and $y = x + 2$ intersect. Substituting the expression for y from the second equation into the first equation, we find that $x + 2 = 4 - x^2$. Rearranging, it follows that

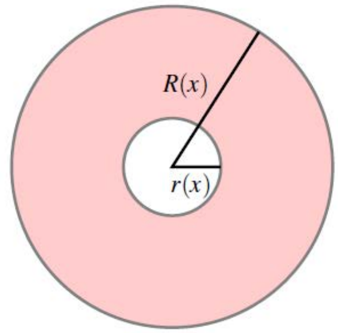
$$x^2 + x - 2 = 0,$$

and the solutions to this equation are $x = -2$ and $x = 1$. The curves therefore intersect at (-2, 0) and (1, 1). When you take the region R that lies between the curves and revolve it about the *x*-axis, we get the three-dimensional solid pictured.









To find the volume of a washer we will subtract the cut-out (inner) disk from the overall outer disk, as follows

$$V_{washer} = V_{outer} - V_{inner}$$

At a given location x between x = -2 and x = 1, the small radius r(x) of the inner hole is determined by the curve y = x + 2, so r(x) = x + 2. Similarly, the big radius R(x) comes from the function y = 4 - x^2, and thus R(x) = 4 - x^2.

To find the volume of a representative slice, you have

$$V_{slice} = \pi[R(x)]^2\Delta\text{x} - \pi[r(x)]^2\ \Delta x$$

Now replace the radii with their values (you can also factor out π and △x),

$$V_{slice} = \pi[(4 - x^2)^2 - (x + 2)^2]\ \Delta x$$

Then, using a definite integral to sum the volumes of the respective slices across the integral, we find that

$$V = \int_{-2}^{1} \pi[(4 - x^2)^2 - (x + 2)^2]\ dx$$

Evaluating the integral, the volume of the solid of revolution is V = 108 u^3.

The general principle to find the volume of a solid of revolution generated by a a region between two curves is often called the washer method.

If $y = r(x)$ and $y = R(x)$ are nonnegative continuous functions on [a,b], that satisfy $R(x) \geq r(x)$ for all x in [a,b], then the volume of the solid of revolution generated by revolving the region between them about the x-axis over this interval is given by

$$V = \int_a^b \pi\{[R(x)]^2 - [r(x)]^2\}\,dx$$

Investigation 3: In each of the following questions, draw a careful, labeled sketch of the region described, as well as the resulting solid that results from revolving the region about the stated axis. In addition, draw a representative slice and state the volume of that slice, along with a definite integral whose value is the volume of the entire solid. It is not necessary to evaluate the integrals you find.

3a) The region S bounded by the x-axis, the curve $y = \sqrt{x}$, and the line $x = 4$; revolve S about the x-axis.

b) The region S bounded by the y-axis, the curve $y = \sqrt{x}$, and the line $y = 2$; revolve S about the x-axis.

c) The finite region S bounded by the curves $y = \sqrt{x}$ and $y = x^3$; revolve S about the x-axis.

d) The finite region S bounded by the curves $y = 2x^2 + 1$ and $y = x^2 + 4$; revolve S about the x-axis

e) The region S bounded by the y-axis, the curve $y = \sqrt{x}$, and the line $y = 2$; revolve S about the y-axis. How does the problem change considerably when we revolve about the y-axis?

To explore more solids of revolution: https://ggbm.at/SSQgqAMT

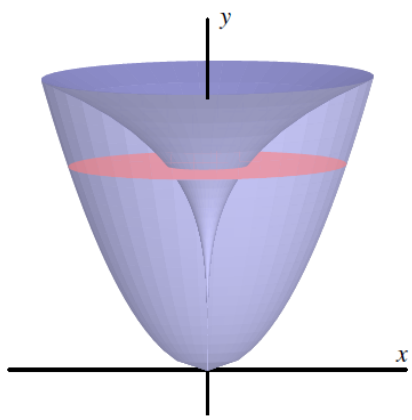
III. Revolving about the y-axis

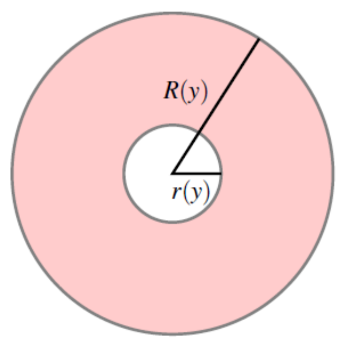
As seen in Investigation 3, problem (e), the problem changes considerably when you revolve a given region about the y-axis. Foremost, this is because representative slices now have thickness △y, which means that it becomes necessary to integrate with respect to y. Let's consider a particular example to demonstrate some of the key issues.

Example 3. Find the volume of the solid of revolution generated when the finite region R that lies between $y = \sqrt{x}$ and $y = x^4$ is revolved about the y-axis.

Solution.

First, find where the two curves intersect when x = 1, hence at the point (1, 1).





Next, notice that the thickness of a representative slice is y, and that the slices are only cylindrical washers in nature when taken perpendicular to the y-axis. Hence, we envision slicing the solid horizontally, starting at y = 0 and proceeding up to y = 1. Because the inner radius is governed by the curve y = x, but from the perspective that x is a function of y, we solve for x and get $x = y^2 = r(y)$. In the same way, we need to view the curve $y = x^4$ (which governs the outer radius) in the form where x is a function of y, and hence $x = \sqrt[4]{y}$. Therefore, we see that the volume of a typical slice is

$$V_{slice} = \pi\{[R(y)]^2 - \pi[r(x)]^2\}\,\Delta y$$

$$= \pi[(\sqrt[4]{y})^2 - (y^2)^2]\,\Delta y$$

Using a definite integral to sum the volume of all the representative slices from y = 0 to y = 1, the total volume is

$$V = \int_{y=0}^{y=1} \pi\left[(\sqrt[4]{y})^2 - (y^2)^2\right]dy$$

It is straightforward to evaluate the integral and find that $V = \frac{7}{15}$ units^3?.

Investigation 4: In each of the following questions, draw a careful, labeled sketch of the region described, as well as the resulting solid that results from revolving the region about the stated axis. In addition, draw a representative slice and state the volume of that slice, along with a definite integral whose value is the volume of the entire solid. It is not necessary to evaluate the integrals you find.

4a) The region S bounded by the y-axis, the curve $y = \sqrt{x}$, and the line y = 2; revolve S about the y-axis.

b) The region S bounded by the x-axis, the curve $y = \sqrt{x}$, and the line x = 4; revolve S about the y-axis.

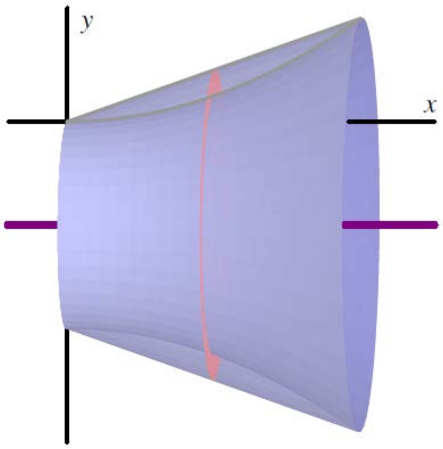
c) The finite region S in the first quadrant bounded by the curves y = 2x and y = x^3; revolve S about the x-axis.

d) The finite region S in the first quadrant bounded by the curves y = 2x and y = x^3; revolve S about the y-axis.

e) The finite region S bounded by the curves x = $(y – 1)^2$ and y = x - 1; revolve S about the y-axis

IV. Revolving about horizontal and vertical lines other than the coordinate axes

Just as you can revolve about one of the coordinate axes (y = 0 or x = 0), it is also possible to revolve around any horizontal or vertical line. Doing so essentially adjusts the radii of cylinders or washers involved by a constant value. A careful, well-labeled plot of the solid of revolution will usually reveal how the different axis of revolution affects the definite integral we set up. Again, an example is instructive.



Example 4. Find the volume of the solid of revolution generated when the finite region S that lies between $y = x^2$ and y = x is revolved about the line y = -1.

Solution.

Graphing the region between the two curves in the first quadrant between their points of intersection (0, 0) and (1, 1) and then revolving the region about the line y = 1, results in the shape aove. Each slice of the solid perpendicular to the axis of revolution is a washer, and the radii of each washer are governed by the curves $y = x^2$ and y = x. But we also see that there is one added change: the axis of revolution adds a fixed length to each radius. In particular, the inner radius of a typical slice, r(x), is given by $r(x) = x^2 + 1$, while the outer radius is R(x) = x + 1. Therefore, the volume of a typical slice is

$$V_{slice} = \pi[R(x)^2 - r(x)^2]\Delta x$$

$$= \pi[(x+1)^2 - (x^2+1)^2]\,\Delta x$$

Finally, we integrate to find the total volume, and

$$V = \int_0^1 \pi[(x+1)^2 - (x^2+1)^2]\ dx = \frac{7\pi}{15}u^3$$

Investigation 5: In each of the following questions, draw a careful, labeled sketch of the region described, as well as the resulting solid that results from revolving the region about the stated axis. In addition, draw a representative slice and state the volume of that slice, along with a definite integral whose value is the volume of the entire solid. It is not necessary to evaluate the integrals you find. For each prompt, use the finite region S in the first quadrant bounded by the curves $y = 2x$ and $y = x^3$.

a) Revolve S about the line $y = -2$.

b) Revolve S about the line $y = 4$.

c) Revolve S about the line $x = -1$.

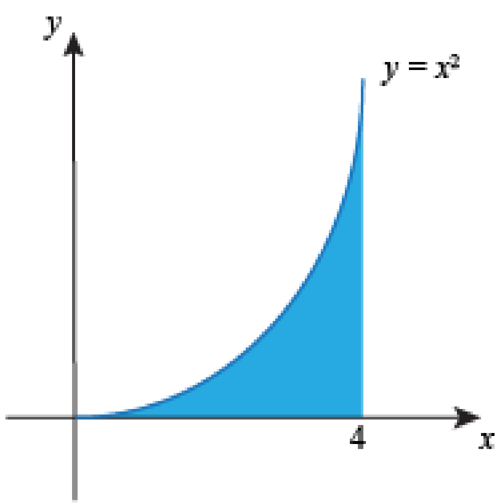
d) Revolve S about the line $x = 5$.

V. Solids with known cross sections

If the area of a known cross section of a solid is known and can be expressed in terms of x, then the volume of a typical slice $\triangle V$ can be determined. The volume of the solid is obtained, as usual, by letting the number of representative slices increase indefinitely.

Example 5

Let R be the region enclosed by the x-axis, the graph $y = x^2$, and the line $x = 4$. Write an integral expression for the volume of the solid whose base is R and whose slices perpendicular to the x-axis are semi-circles.



Solution.

This solid is like the one in the example except that instead of the slices being squares, they're semicircles. The slice at position x is a semi-circle with diameter $y = x^2$ and thickness Δx.

This solid really looks like a loaf of French bread. Since the diameter of a slice is x^2, the radius is $\frac{x^2}{2}$ The volume of a slice is the area of the semi-circle (half the area of a circle) multiplied by the thickness, or

$$V_{slice} = \frac{1}{2}\pi\left(\frac{x^2}{2}\right)^2 \Delta x$$

The variable x goes from 0 to 4 in this region, so when we take the integral we get

$$V = \int_0^4 \frac{1}{2}\pi\left(\frac{x^2}{2}\right)^2 dx$$

VI. Practice

1. Consider the curve $f(x) = 3\cos\left(\frac{x^3}{4}\right)$ and the portion of its graph that lies in the first quadrant between the y-axis and the first positive value of x for which f(x) = 0. Let R denote the region bounded by this portion of f , the x-axis, and the y-axis.

a) Set up a definite integral whose value is the exact arc length of f that lies along the upper boundary of R. Use technology appropriately to evaluate the integral you find.

b) Set up a definite integral whose value is the exact area of R. Use technology appropriately to evaluate the integral you find.

c) Suppose that the region R is revolved around the x-axis. Set up a definite integral whose value is the exact volume of the solid of revolution that is generated. Use technology appropriately to evaluate the integral you find.

d) Suppose instead that R is revolved around the y-axis. If possible, set up an integral expression whose value is the exact volume of the solid of revolution and evaluate the integral using appropriate technology. If not possible, explain why.

2. Consider the curves given by y = sin x and y = cos x . For each of the following problems, you should include a sketch of the region/solid being considered, as well as a labeled representative slice.

a) Sketch the region R bounded by the y-axis and the curves y = sin x and y = cos x up to the first positive value of x at which they intersect. What is the exact intersection point of the curves?

b) Set up a definite integral whose value is the exact area of R.

c) Set up a definite integral whose value is the exact volume of the solid of revolution generated by revolving R about the x-axis.

d) Set up a definite integral whose value is the exact volume of the solid of revolution generated by revolving R about the y-axis.

e) Set up a definite integral whose value is the exact volume of the solid of revolution generated by revolving R about the line y = 2.

f) Set up a definite integral whose value is the exact volume of the solid of revolution generated by revolving R about the x = -1.

3. Consider the finite region R that is bounded by the curves $y = 1 + \frac{1}{2}(x-2)^2$, $y = \frac{1}{2}x^2$ and x = 0.

a) Determine a definite integral whose value is the area of the region enclosed by the two curves.

b) Find an expression involving one or more definite integrals whose value is the volume of the solid of revolution generated by revolving the region R about the line y = -1.

c) Determine an expression involving one or more definite integrals whose value is the volume of the solid of revolution generated by revolving the region R about the y-axis.

d) Find an expression involving one or more definite integrals whose value is the perimeter of the region R.

4. Let R be the region bounded by $y = x$ and $y = x^2$. Write an integral expression for the volume of the solid with base R whose slices perpendicular to the y-axis are squares

VII. Practice – Khan Academy

1. Complete the first three online practice exercises in the Volume Using Calculus unit of Khan Academy's AP Calculus AB course: https://www.khanacademy.org/math/ap-calculus-ab/volume-using-calculus-ab?t=practice