3.53 Integral Area

Using Definite Integrals to Find Area and Length
Calculus

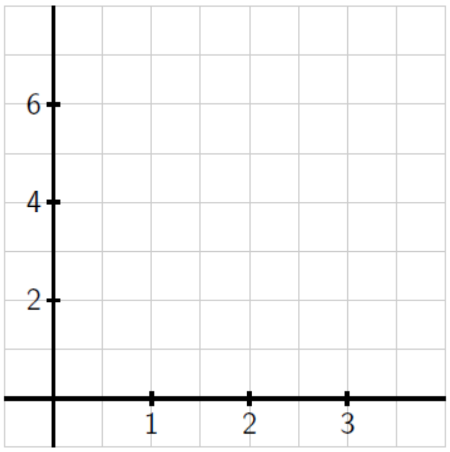
Early in your work with the definite integral, you learned that if you have a nonnegative velocity function, v, for an object moving along an axis, the area under the velocity function between a and b tells you the total distance the object traveled on that time interval. Moreover, based on the definition of the definite integral, that area is given precisely by $\int_a^b v(t)\, dt$. Indeed, for any nonnegative function f on an interval $[a, b]$, we know that $\int_a^b f(x)\, dx$ measures the area bounded by the curve and the x-axis between $x = a$ and $x = b$.

In this lesson, you will explore how definite integrals can be used to represent a variety of different physically important properties. In Investigation1, you will see how a single definite integral may be used to represent the area between two curves.

Investigation1: Consider the functions given by $f(x) = 5 - (x - 1)^2$ and $g(x) = 4 - x$.

a) Use algebra to find the points where the graphs of f and g intersect.

b) Sketch an accurate graph of f and g on the axes provided, labeling the curves by name and the intersection points with ordered pairs.



c) Find and evaluate exactly an integral expression that represents the area between $y = f(x)$ and the x-axis on the interval between the intersection points of f and g.

d) Find and evaluate exactly an integral expression that represents the area between $y = g(x)$ and the x-axis on the interval between the intersection points of f and g.

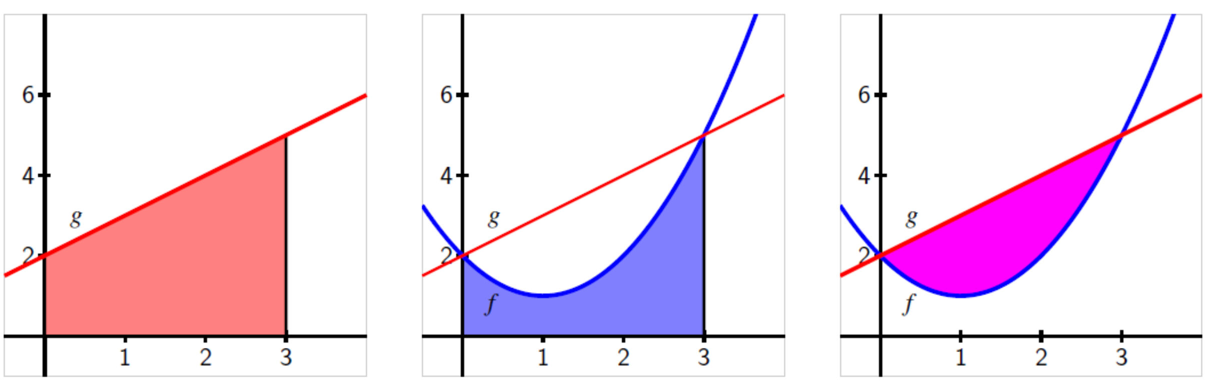
e) What is the exact area between f and g between their intersection points? Why?

II. The Area Between Two Curves

In Investigation 1, you saw a natural way to think about the area between two curves: the area beneath the upper curve minus the area below the lower curve. For the functions $f(x) = (x - 1)^2 + 1$ and $g(x) = x + 2$, shown below, we see that the upper curve is $g(x) = x + 2$, and that the graphs intersect at $(0, 2)$ and $(3, 5)$. Note that we can find these intersection points by solving the system of equations given by $y = (x - 1)^2 + 1$ and $y = x + 2$ through substitution: substituting $x + 2$ for y in the first equation yields $x + 2 = (x - 1)^2 + 1$, so $x + 2 = x^2 - 2x + 1 + 1$, and thus

$$x^2 - 3x = x(x - 3) = 0,$$

from which it follows that $x = 0$ or $x = 3$. Using $y = x + 2$, we find the corresponding y-values of the intersection points.



On the interval [0, 3], the area beneath g is

$$\int_0^3 (x + 2)dx = \frac{21}{2}$$

while the area under f on the same interval is

$$\int_0^3 [(x-1)^2+1]dx = 6$$

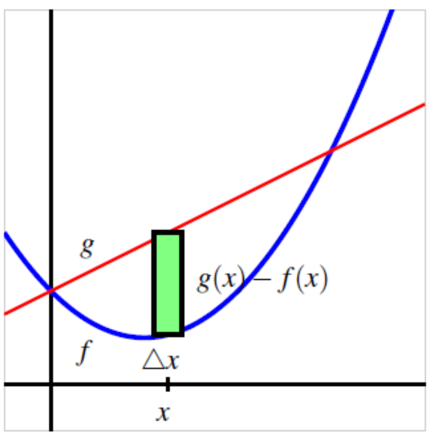
Thus, the area between the curves is

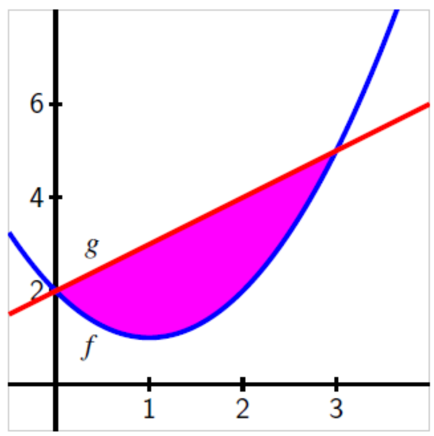
$$A = \int_0^3 (x+2)dx - \int_0^3 [(x-1)^2+1]dx$$

$$A = \frac{21}{2} - 6 = \frac{9}{2}\text{units}^2$$

A slightly different perspective is also helpful here: if you take the region between two curves and slice it up into thin vertical rectangles, then the height of a typical rectangle is given by the difference between the two functions. For example, for the rectangle shown below left, the rectangle's height is $g(x)$ - $f(x)$, while its width can be viewed as $\triangle x$, and thus the area of the rectangle is

$$A_{\text{rect}} = (g(x) - f(x))\, \triangle x.$$





The area between the two curves on [0, 3] is thus approximated by the Riemann sum

$$A \approx \sum_{i=1}^{n} [(g(x_i) - f(x_i)]\Delta x$$

and when $n \to \infty$, it follows that the area is given by the single definite integral

$$A = \int_0^3 [(g(x) - f(x)]dx$$

In many applications of the definite integral, you will find it helpful to think of a "representative slice" and how the definite integral may be used to combine these slices to find the exact value of a desired quantity. Here, the integral essentially sums the areas of infinitely thin rectangles.

Finally, whether we think of the area between two curves as the difference between the area bounded by the individual curves or as the limit of a Riemann sum that adds the areas of thin rectangles between the curves, these two results are the same, since the difference of two integrals is the integral of the difference:

$$\int_0^3 g(x)\,dx - \int_0^3 f(x)\,dx = \int_0^3 [(g(x) - f(x)]dx$$

This exemplifies the following general principle.

If two curves $y = g(x)$ and $y = f(x)$ intersect at $(a, g(a))$ and $(b, g(b))$, for all x such that $a \leq x \leq b$, $g(x) \geq f(x)$, then the area between the curves is

$$A = \int_a^b [g(x) - f(x)]dx$$

Investigation 2: In each of the following problems, the goal is to determine the area of the region described. For each region,

(i) determine the intersection points of the curves,

(ii) sketch the region whose area is being found,

(iii) draw and label a representative slice, and

(iv) state the area of the representative slice.

(v) Then, state a definite integral whose value is the exact area of the region, and evaluate the integral to find the numeric value of the region's area.

2a) The finite region bounded by $y = \sqrt{x}$ and $y = \frac{1}{4}x$.

(b) The finite region bounded by $y = 12 - 2x^2$ and $y = x^2 - 8$.

(c) The area bounded by the y-axis, $f(x) = \cos(x)$, and $g(x) = \sin(x)$, where we consider the region formed by the first positive value of x for which f and g intersect.

(d) The finite regions between the curves $y = x^3 - x$ and $y = x^2$.

III. Finding Area with Horizontal Slices

At times, the shape of a geometric region may dictate that we need to use horizontal rectangular slices, rather than vertical ones. For instance, consider the region bounded by the parabola $x = y^2 - 1$ and the line $y = x - 1$, pictured below. First, observe that by solving the second equation for x and writing $x = y + 1$, you can eliminate a variable through substitution and find that $y + 1 = y^2 - 1$, and hence the curves intersect where $y^2 - y - 2 = 0$. Thus, you find $y = -1$ or $y = 2$, so the intersection points of the two curves are (0, -1) and (3, 2).

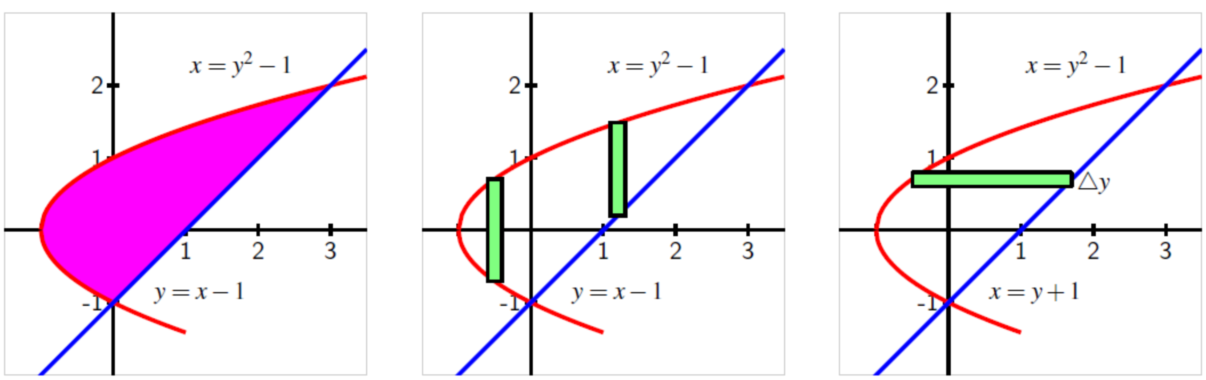
If you attempt to use vertical rectangles to slice up the area, at certain values of x (specifically from $x = 1$ to $x = 0$, as seen in the center graph), the curves that govern the top and bottom of the rectangle are one and the same.

Instead, as shown in the rightmost graph, use horizontal rectangles to compute the area of the region. Notice that the width depends on y, the height at which the rectangle is constructed. In particular, at a height y between $y = 1$ and $y = 2$, the right end of a representative rectangle is determined by the line, $x = y + 1$, while the left end of the rectangle is determined by the parabola, $x = y^2 - 1$, and the thickness of the rectangle is $\triangle y$. Therefore the area of the rectangle is

$$A_{rect} = [(y+1) - (\qquad - 1)] \triangle \mathrm{y}$$

and the area of the area between the two curves on the y-interval [-1, 2] is

$$A = \int_{y=-1}^{y=2} [\big((y+1) - (y^2 - 1)\big)]dy$$



Notice that you are now integrating with respect to y; this is dictated by the fact that you chose to use horizontal rectangles whose widths depend on y and whose thickness is denoted $\triangle y$. Evaluate and you get $A = \frac{9}{2}$ units².

Just as with the use of vertical rectangles of thickness $\triangle x$, there is a general principle for finding the area between two curves, which follows.

> If two curves $y = g(x)$ and $y = f(x)$ intersect at $(g(a)$,c) , for all x such that $c \leq y \leq d,\ g(y) \geq f(y)$, then the area between the curves is
>
> $$A = \int_{y=c}^{y=d} [g(y) - f(y)]dy$$

Investigation 3: In each of the following problems, our goal is to determine the area of the region described. For each region,

(i) determine the intersection points of the curves,

(ii) sketch the region whose area is being found,
(iii) draw and label a representative slice,
(iv) state the area of the representative slice, and
(v) state a definite integral whose value is the exact area of the region, and evaluate the integral to find the numeric value of the region's area.

Note well: At the step where you draw a representative slice, you need to make a choice about whether to slice vertically or horizontally.

3a) The finite region bounded by $x = y^2$ and $x = 6 - 2y^2$.

b) The finite region bounded by $x = 1 - y^2$ and $x = 2 - 2y^2$.

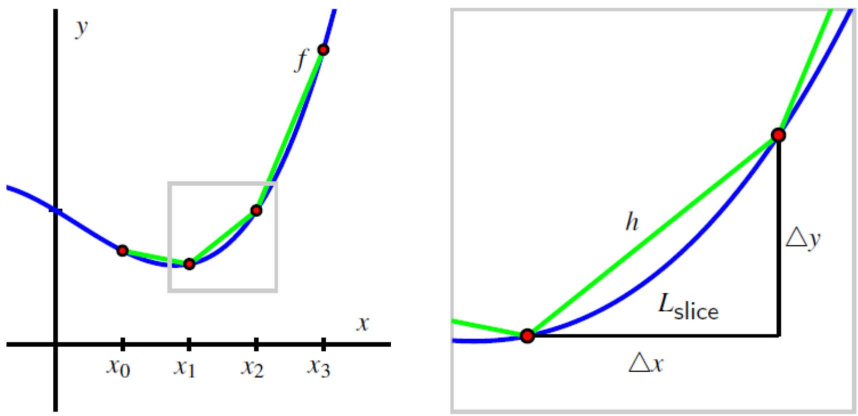
c) The area bounded by the x-axis, $y = x^2$, and $y = 2 - x$.

d) The finite regions between the curves $x = y^2 - 2y$ and $y = x$.

e. In your own words, how do you decide whether to slice vertically or horizontally?

IV. Finding the length of a curve

The definite integral can also be used to find the length of a portion of a curve. You will use the same fundamental principle: take a curve and slice it up into small pieces whose lengths can be approximated.



To compute the arc length of the curve $y = f(x)$ from $x = a$ to $x = b$, think about the portion of length, L_{slice}, that lies along the curve on a small interval of length $\triangle x$, and estimate the value of L_{slice} using a well-chosen triangle. Consider the right triangle with legs parallel to the coordinate axes and hypotenuse connecting two points on the curve, as seen enlarged above right.

$$L_{slice} \approx h = \sqrt{(\triangle \text{x})^2 + (\triangle \text{y})^2}$$

By algebraically rearranging the expression for the length of the hypotenuse, the definite integral can be used to compute the length of a curve. In particular, observe that by removing a factor of $(\triangle x)^2$, you get

$$= \sqrt{(\triangle \text{x})^2 (1 + \frac{(\triangle \text{y})^2}{(\triangle \text{x})^2})}$$

$$= \sqrt{(1 + \frac{(\triangle \text{y})^2}{(\triangle \text{x})^2})} \cdot \Delta x$$

Furthermore, as $n \to \infty$ and $\Delta x \to 0$, it follows that $\frac{\Delta y}{\Delta x} \to \frac{dy}{dx} = f'(x)$. Thus, one can say that

$$L_{slice} = \sqrt{1 + f'(x)^2} \cdot \Delta x$$

Taking a Riemann sum of the slices and letting $n \to \infty$, you arrive at the following fact.

> Given a differentiable function f on an interval $[\, a, b\,]$ the total arc length, L, of the curve $y = f(x)$ from $x = a$ to $x = b$ is given by
>
> $$L = \int_a^b \sqrt{1 + f'(x)^2} \cdot dx$$

Investigation 4: Each of the following questions somehow involves the arc length along a curve.

4a) Use the definition and appropriate computational technology to determine the arc length along $y = x^2$ from $x = -1$ to $x = 1$.

b) Find the arc length of $y = \sqrt{4 - x^2}$ on the interval $-2 \leq x \leq 2$. Find this value in two different ways: (i) by using a definite integral, and (ii) by using a familiar property of the curve.

c) Determine the arc length of $y = xe^{3x}$ on the interval $[0, 1]$.

d) Will the integrals that arise calculating arc length typically be ones that we can evaluate exactly using the First FTC, or ones that we need to approximate? Why?

e) A moving particle is traveling along the curve given by $y = f(x) = 0.1x^2 + 1$, and does so at a constant rate of 7 cm/sec, where both x and y are measured in cm (that is, the curve by $y = f(x)$ is the path along which the object actually travels; the curve is not a "position function"). Find the position of the particle when $t = 4$ sec, assuming that when t = 0, the particle's location is $(0, f(0))$.

IV Practice

1. Find the exact area of each described region.

   a) The finite region between the curves $x = y(y - 2)$ and $x = -(y - 1)(y - 3)$.

   b) The region between the sine and cosine functions on the interval $[\frac{\pi}{4}, \frac{3\pi}{4}]$

   c) The finite region between $x = y^2 - y - 2$ and $y = 2x - 1$.

   d) The finite region between $y = mx$ and $y = x^2 - 1$, where m is a positive constant.

2. Let $f(x) = 1 - x^2$ and $g(x) = ax^2 - a$, where a is an unknown positive real number. For what value(s) of a is the area between the curves f and g equal to 2?

3. Let $f(x) = 2 - x^2$. Recall that the average value of any continuous function f on an interval [a, b] is given by $\frac{1}{b-a}\int_a^b f(x)dx$.

a) Find the average value of $f(x) = 2 - x^2$ on the interval $[0, \sqrt{2}]$. Call this value r.

b) Sketch a graph of $y = f(x)$ and $y = r$. Find their intersection point(s).

c) Show that on the interval $[0, \sqrt{2}]$, the amount of area that lies below $y = f(x)$ and above $y = r$ is equal to the amount of area that lies below $y = r$ and above $y = f(x)$.

d) Will the result of (c) be true for any continuous function and its average value on any interval? Why?

V. Assessment – Khan Academy

1. Complete four online practice exercises (#1, 2, 4, 5) in the *Integration Applications* unit of Khan Academy's AP Calculus AB course: https://www.khanacademy.org/math/ap-calculus-ab/integration-applications-ab?t=practice