3.4e It's a Wild, Wild World

Population Growth and Logistic Equations

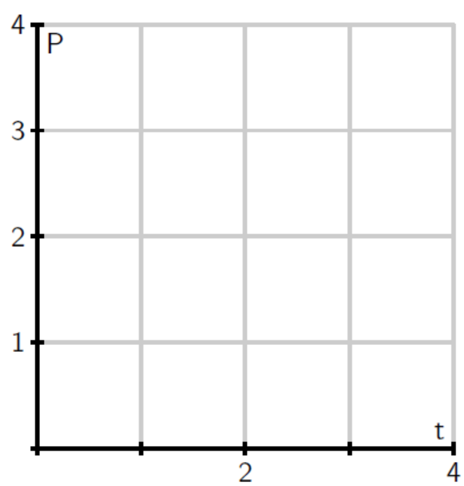
The growth of the earth's population is one of the pressing issues of our time. Will the population continue to grow? Or will it perhaps level off at some point, and if so, when? In this section, you will look at two ways in which you may use differential equations to help you address questions such as these.

Investigation 1: Recall that one model for population growth states that a population grows at a rate proportional to its size.

1. You begin with the differential equation

$$\frac{dP}{dt} = \frac{1}{2}P$$

a) Sketch a slope field below as well as a few typical solutions on the axes provided.



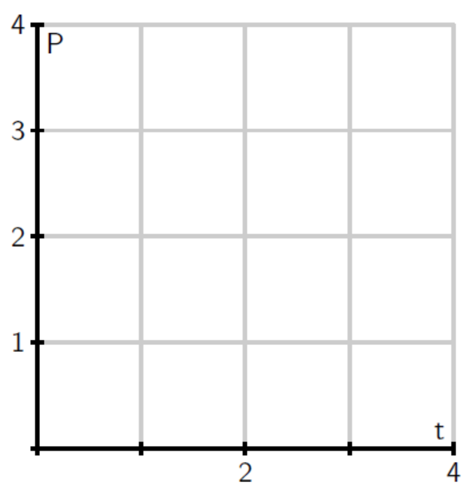
b) Find all equilibrium solutions of the equation $\frac{dP}{dt} = \frac{1}{2}P$ and classify them as stable or unstable.

c) If $P(0)$ is positive, describe the long-term behavior of the solution to $\frac{dP}{dt} = \frac{1}{2}P$.

d) Now consider a modified differential equation given by

$$\frac{dP}{dt} = \frac{1}{2}P(3 - P)$$

As before, sketch a slope field as well as a few typical solutions on the following axes provided.



e) Find any equilibrium solutions and classify them as stable or unstable.

f) If $P(0)$ is positive, describe the long-term behavior of the solution.

II. The earth's population

You will now begin studying the earth's population. To get started, here are some data for the earth's population in recent years that you will use in your investigations.

Year	Population (in billions)
1998	5.932
1999	6.008
2000	6.084
2001	6.159
2002	6.234
2005	6.456
2006	6.531
2007	6.606
2008	6.681
2009	6.756
2010	6.831

Investigation 2: Your first model will be based on the following assumption:

The rate of change of the population is proportional to the population.

On the face of it, this seems reasonable. When there is a relatively small number of people, there will be fewer births and deaths so the rate of change will be small. When there is a larger number of people, there will be more births and deaths so you expect a larger rate of change.

2. If $P(t)$ is the population t years after the year 2000, we may express this assumption as

$$\frac{dP}{dt} = kP$$

where k is a constant of proportionality.

a) Use the data in the table to estimate the derivative $P'(0)$ using a central difference. Assume that $t = 0$ corresponds to the year 2000.

b) What is the population $P(0)$?

c) Use these two facts to estimate the constant of proportionality k in the differential equation.

d) Now that you know the value of k, you have the initial value problem

$$\frac{dP}{dt} = kP, P(0) = 6.084$$

Find the solution to this initial value problem.

e) What does your solution predict for the population in the year 2010? Is this close to the actual population given in the table?

f) When does your solution predict that the population will reach 12 billion?

g) What does your solution predict for the population in the year 2500?

h) Do you think this is a reasonable model for the earth's population? Why or why not? Explain your thinking using a couple of complete sentences.

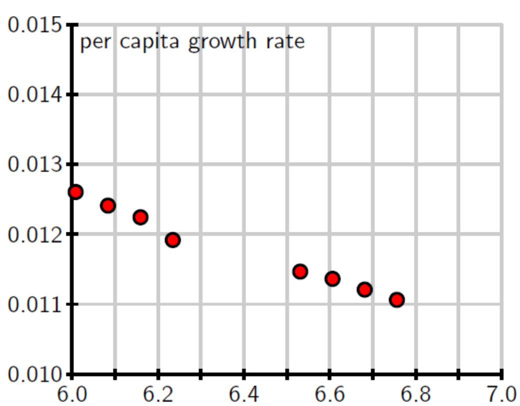
Your work in Investigation 2 shows that that the exponential model is fairly accurate for years relatively close to 2000. However, if you go too far into the future, the model predicts increasingly large rates of change, which causes the population to grow arbitrarily large. This does not make much sense since it is unrealistic to expect that the earth would be able to support such a large population.

The constant k in the differential equation has an important interpretation. Let's rewrite the differential equation $\frac{dP}{dt} = kP$ by solving for k, so that you have

$$k = \frac{dP/dt}{P}$$

Viewed in this light, k is the ratio of the rate of change to the population; in other words, it is the contribution to the rate of change from a single person. You call this the ***per capita growth rate***.

In the exponential model introduced in Investigation 2, the per capita growth rate is constant. In particular, you are assuming that when the population is large, the per capita growth rate is the same as when the population is small. It is natural to think that the per capita growth rate should decrease when the population becomes large, since there will not be enough resources to support so many people. In other words, you expect that a more realistic model would hold if you assume that the per capita growth rate depends on the population P.



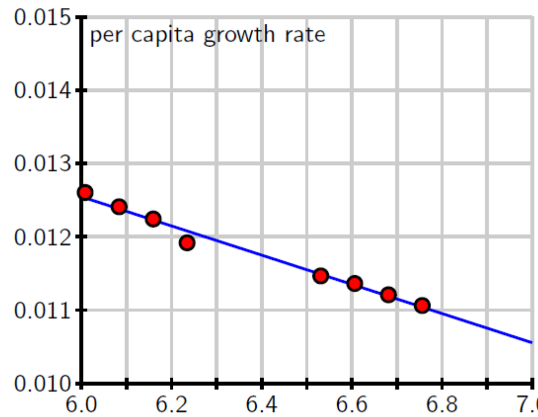
In the previous activity, you computed the per capita growth rate in a single year by computing k, the quotient of $\frac{dP}{dt}$ and P (which you did for $t = 0$).

If you return data and compute the per capita growth rate over a range of years, you generate the data shown at left, which shows how the per capita growth rate is a function of the population, P.

From the data, notice that the per capita growth rate appears to decrease as the population increases. In fact, the points seem to lie very close to a line, which is shown at two different scales in the figures below.

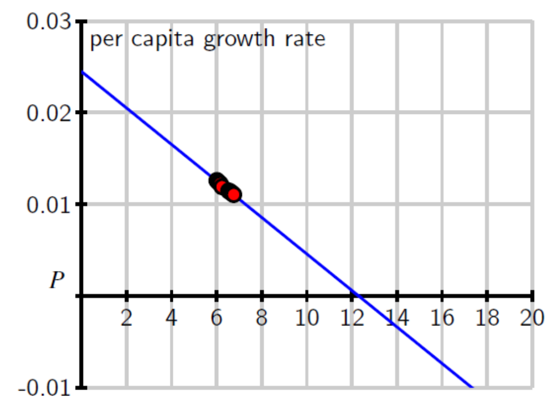
Looking at this line carefully, you can find its equation to be

$$\frac{dP/dt}{P} = 0.025 - 0.002P$$



If you multiply both sides by P, you arrive at the differential equation

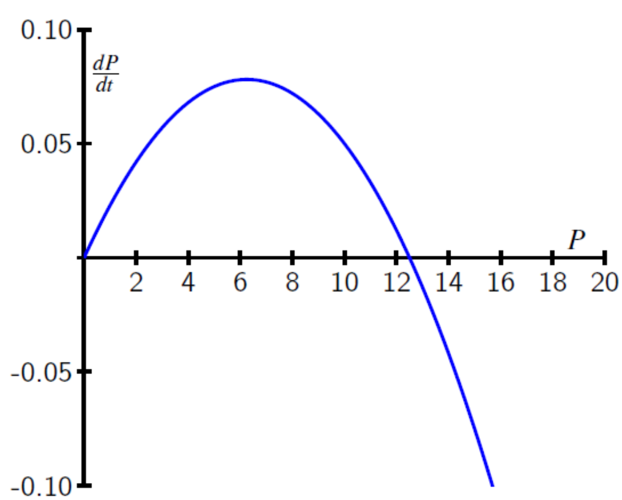
$$\frac{dP}{dT} = P(0.025 - 0.002P)$$



Graphing the dependence of $\frac{dP}{dT}$ on the population P, you see that this differential equation demonstrates a quadratic relationship between $\frac{dP}{dT}$ and P, as shown in at right.

The equation $\frac{dP}{dT} = P(0.025 - 0.002P)$ is an example of the ***logistic equation***, and is the second model for population growth that you will consider. You have reason to believe that it will be more realistic since the per capita growth rate is a decreasing function of the population.

Indeed, the graph at left, shows that there are two equilibrium solutions, $P = 0$, which is unstable, and $P = 12.5$, which is a stable equilibrium. The graph shows that any solution with $P(0) > 0$ will eventually stabilize around 12.5. In other words, your model predicts the world's population will eventually stabilize around 12.5 billion.



A prediction for the long-term behavior of the population is a valuable conclusion to draw from your differential equation. You would, however, like to answer some quantitative questions. For instance, how long will it take to reach a population of 10 billion? To determine this, you need to find an explicit solution of the equation.

Solving the logistic differential equation

To apply the logistic model in more general situations, first state the logistic equation in its more general form,

$$\frac{dP}{dT} = kP(N - P)$$

The equilibrium solutions here are when P = 0 and $1 - \frac{P}{N} = 0$, which shows that $P = N$. The equilibrium at $P = N$ is called the ***carrying capacity*** of the population for it represents the stable population that can be sustained by the environment.

Now solve the logistic equation. The equation is separable, so you separate the variables

$$\frac{1}{P(N - P)}\frac{dP}{dt} = k,$$

and integrate to find that

$$\int \frac{1}{P(N - P)}\,dP = \int k\,dt$$

To find the antiderivative on the left, you use the partial fraction decomposition

$$\frac{1}{P(N - P)} = \frac{1}{N}\left[\frac{1}{P} + \frac{1}{P(N - P)}\right]$$

Now you are ready to integrate, with

$$\int \frac{1}{N}\left[\frac{1}{P} + \frac{1}{P(N - P)}\right]dP = \int k\,dt$$

On the left, observe that N is constant, so you can remove the factor of 1 and antidifferentiate to find that

$$\frac{1}{N}(\ln|P| - \ln|N - P|) = kt + C$$

Multiplying both sides of this last equation by N and using an important rule of logarithms, you next find that

$$\ln\left|\frac{P}{N - P}\right| = kNt + C$$

From the definition of the logarithm, replacing e^C with C, and letting C absorb the absolute value signs, you now know that

$$\frac{P}{N-P} = Ce^{kNt}$$

At this point, all that remains is to determine C and solve algebraically for P. If the initial population is $P(0) = P_0$, then it follows that $C = \frac{P_0}{N-P_0}$, so

$$\frac{P}{N-P} = \frac{P_0}{N-P_0}e^{kNt}$$

You will solve this most recent equation for P by multiplying both sides by $(N-P)(N-P_0)$ to obtain

$$P(N-P_0) = P_0(N-P)e^{kNt}$$

$$P(N-P_0) = P_0Ne^{kNt} - P_0Pe^{kNt}.$$

Swapping the left and right sides, expanding, and factoring, it follows that

$$P_0Ne^{kNt} = P(N-P_0) + P_0Pe^{kNt}$$

$$= P(N-P_0+P_0Pe^{kNt}).$$

Dividing to solve for P, you see that

$$P = \frac{P_0Ne^{kNt}}{N-P_0+P_0Pe^{kNt}}$$

Finally, you choose to multiply the numerator and denominator by $\frac{1}{P_0}e^{-kNt}$ to obtain

$$P = \frac{N}{\left(\frac{N-P_0}{P_0}\right)e^{-kNt}+1}$$

While that was a lot of algebra, notice the result: you have found an explicit solution to the initial value problem $\frac{dP}{dT} = kP(N-P),\ P(0) = P_0$ and that solution is

$$P(t) = \frac{N}{\left(\frac{N-P_0}{P_0}\right)e^{-kNt}+1}$$

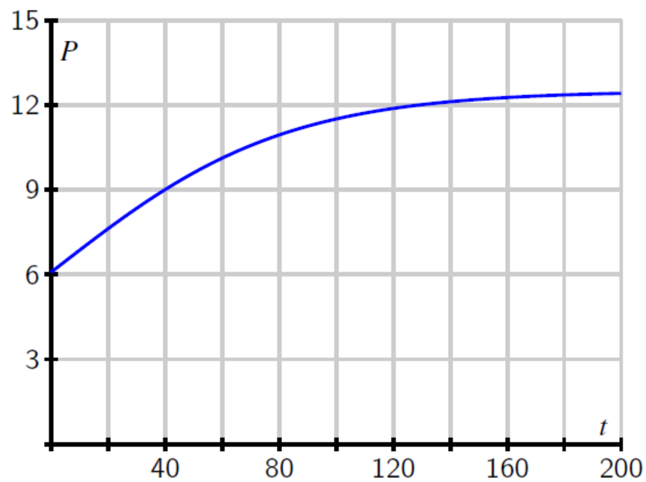
For the logistic equation describing the earth's population that you worked with earlier, you have

$k = 0.002, \quad N = 12.5, \quad \text{and} \quad P(0) = 6.084.$

This gives the solution

$$P(t) = \frac{12.5}{1.0546\, e^{-0.025t} + 1}$$

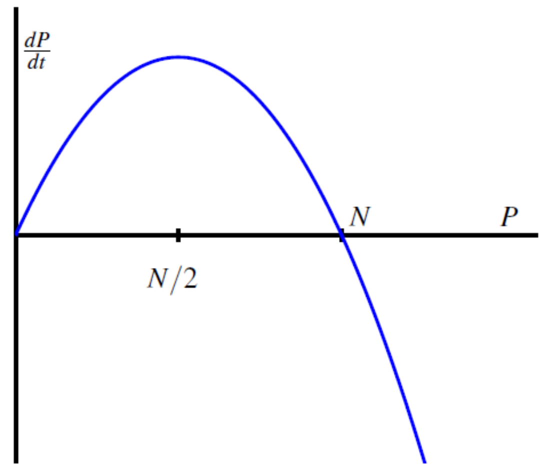
whose graph is shown below Notice that the graph shows the population leveling off at 12.5 billion, as you expected, and that the population will be around 10 billion in the year 2050. These results, which you have found using a relatively simple mathematical model, agree fairly well with predictions made using a much more sophisticated model developed by the United Nations.



The logistic equation is useful in other situations, too, as it is good for modeling any situation in which limited growth is possible. For instance, it could model the spread of a flu virus through a population contained on a cruise ship, the rate at which a rumor spreads within a small town, or the behavior of an animal population on an island. Again, it is important to realize that through your work in this section, you have completely solved the logistic equation, regardless of the values of the constants N, k, and P_0. Anytime you encounter a logistic equation, you can apply the formula you used here!

$$\frac{dP}{dT} = kP(N - P)$$

Investigation 3: Consider the logistic equation $\frac{dP}{dT} = kP(N-P)$ with the graph of $\frac{dP}{dT}$ vs. P shown below.



a) At what value of P is the rate of change greatest?

b) Consider the model for the earth's population that you created. At what value of P is the rate of change greatest? How does that compare to the population in recent years?

c) According to the model you developed, what will the population be in the year 2100?

d) According to the model you developed, when will the population reach 9 billion?

e) Now consider the general solution to the general logistic initial value problem that you found, given by

$$P(t) = \frac{N}{\left(\frac{N-P_0}{P_0}\right)e^{-kNt} + 1}$$

Verify algebraically that $P(0) = P_0$ and that $\lim_{t\to\infty} P(t) = N$.

III. Exercises

1. The logistic equation may be used to model how a rumor spreads through a group of people. Suppose that $p(t)$ is the fraction of people that have heard the rumor on day *t*. The equation

$$\frac{dp}{dt} = 0.2p(1-p)$$

describes how p changes. Suppose initially that one-tenth of the people have heard the rumor; that is, $p(0) = 0.1$.

a) What happens to $p(t)$ after a very long time?

b) Determine a formula for the function $p(t)$.

c) At what time is p changing most rapidly?

d) How long does it take before 80% of the people have heard the rumor?

2. Suppose that $b(t)$ measures the number of bacteria living in a colony in a Petri dish, where b is measured in thousands and t is measured in days. One day, you measure that there are 6,000 bacteria and the per capita growth rate is 3. A few days later, you measure that there are 9,000 bacteria and the per capita growth rate is 2.

a) Assume that the per capita growth rate $\frac{db/dt}{b}$ is a linear function of b. Use the measurements to find this function and write a logistic equation to describe $\frac{db}{dt}$.

b) What is the carrying capacity for the bacteria?

c) At what population is the number of bacteria increasing most rapidly?

d) If there are initially 1,000 bacteria, how long will it take to reach 80% of the carrying capacity?

3. Suppose that the population of a species of fish is controlled by the logistic equation

$$\frac{dP}{dt} = 0.1P(10 - P)$$

where P is measured in thousands of fish and t is measured in years.

a) What is the carrying capacity of this population?

b) Suppose that a long time has passed and that the fish population is stable at the carrying capacity. At this time, humans begin harvesting 20% of the fish every year. Modify the differential equation by adding a term to incorporate the harvesting of fish.

c) What is the new carrying capacity?

d) What will the fish population be one year after the harvesting begins?

e) How long will it take for the population to be within 10% of the carrying capacity?

4. A 2000-gallon tank can support no more than 150 guppies. Six guppies are introduced into the tank. Assume that the rate of growth of the population is

$$\frac{dP}{dt} = 0.0015P(150 - P)$$

where time t is in weeks.

a) Find a formula for the guppy population in terms of t.

b) How long will it take for the guppy population to be 100? 125?

5. A certain wild animal preserve can support no more than 250 lowland gorillas. Twenty-eight gorillas were known to be in the preserve in 1970. Assume that the rate of growth of the population is

$$\frac{dp}{dt} = 0.0004p(250 - p)$$

where time t is in years.

(a) Find a formula for the gorilla population in terms of t.

(b) How long will it take for the gorilla population to reach the carrying capacity of the preserve?

6. The spread of a disease through a community can be modeled with the logistic equation

$$\frac{dy}{dt} = \frac{0.9}{1 + 45e^{-0.15t}}$$

where y is the proportion of people infected after t days. According to the model, what percentage of the people in the community will not become infected?

(A) 2% (B) 10% (C) 15% (D) 45% (E) 90%