3.4e In the Real World

Modeling with Differential Equations

In previous lessons, you observed several ways that differential equations arise in the natural world, from the growth of a population to the temperature of a cup of coffee. In this section, you will look more closely at how differential equations give you a natural way to describe various phenomena.

Investigation 1: Any time that the rate of change of a quantity is related to the amount of a quantity, a differential equation naturally arises. In the following problems, you want to develop a differential equation whose solution is the quantity of interest.

1a) Suppose you have a bank account in which money grows at an annual rate of 3%.

i. If you have $10,000 in the account, at what rate is your money growing?

ii. Suppose that you are also withdrawing money from the account at $1,000 per year. What is the rate of change in the amount of money in the account? What are the units on this rate of change?

1b) Suppose that a water tank holds 100 gallons and that a salty solution, which contains 20 grams of salt in every gallon, enters the tank at 2 gallons per minute.

i. How much salt enters the tank each minute?

ii. Suppose that initially there are 300 grams of salt in the tank. How much salt is in each gallon at this point in time?

iii. Finally, suppose that evenly mixed solution is pumped out of the tank at the rate of 2 gallons per minute. How much salt leaves the tank each minute?

iv. What is the total rate of change in the amount of salt in the tank?

II. Developing a differential equation

Investigation 1 demonstrates the kind of thinking you will be doing in this section. In that example, you considered if there is a quantity, such as the amount of money in the bank account, that is changing due. The governing differential equation results from the total rate of change being the difference between the rate of increase and the rate of decrease.

Example 2: In the Great Lakes region, rivers flowing into the lakes carry a great deal of pollution in the form of small pieces of plastic averaging 1 millimeter in diameter. To understand how the amount of plastic in Lake Michigan is changing, construct a model for how this type pollution has built up in the lake.

Solution.

First, some basic facts about Lake Michigan.

- The volume of the lake is $5 \cdot 10^{12}$ cubic meters.
- Water flows into the lake at a rate of $5 \cdot 10^{10}$ cubic meters per year. It flows out of the lake at the same rate.
- Each cubic meter flowing into the lake contains roughly $3 \cdot 10^{-8}$ cubic meters of plastic pollution.

Let's denote the amount of pollution in the lake by $P(t)$, where P is measured in cubic meters of plastic and t in years. Your goal is to describe the rate of change of this function; in other words, you want to develop a differential equation describing $P(t)$.

First, you will measure how $P(t)$ increases due to pollution flowing into the lake. You know that $5 \cdot 10^{10}$ cubic meters of water enters the lake every year and each cubic meter of water contains $3 \cdot 10^{-8}$ cubic meters of pollution. Therefore, pollution enters the lake at the rate of

$$\left(5 \cdot 10^{10} \frac{m^3 \text{ water}}{\text{year}}\right) \cdot \left(3 \cdot 10^{-8} \frac{m^3 \text{ plastic}}{m^3 \text{ water}}\right) = 1.5 \cdot 10^3 \frac{m^3 \text{ plastic}}{\text{year}}$$

Second, you will measure how $P(t)$ decreases due to pollution flowing out of the lake. If the total amount of pollution is P cubic meters and the volume of Lake Michigan is $5 \cdot 10^{12}$cubic meters, then the concentration of plastic pollution in Lake Michigan is

$$\frac{P}{5 \cdot 10^{12}} \frac{m^3 \text{ plastic}}{m^3 \text{ water}}$$

Since $5 \cdot 10^{10}$ cubic meters of water flow out each year[1], then the plastic pollution leaves the lake at the rate of

$$\left(\frac{P}{5 \cdot 10^{12}} \frac{m^3 \text{ plastic}}{m^3 \text{ water}}\right) \cdot \left(5 \cdot 10^{10} \frac{m^3 \text{ water}}{\text{year}}\right) = \frac{P}{100} \text{ cubic meters of plastic per year.}$$

The total rate of change of P is thus the difference between the rate at which pollution enters the lake minus the rate at which pollution leaves the lake; that is,

$$\frac{dP}{dt} = 1.5 \cdot 10^3 - \frac{P}{100}$$

$$= \frac{1}{100}(1.5 \cdot 10^5).$$

You have now found a differential equation that describes the rate at which the amount of pollution is changing. To better understand the behavior of $P(t)$, you now apply some of the techniques you have recently developed.

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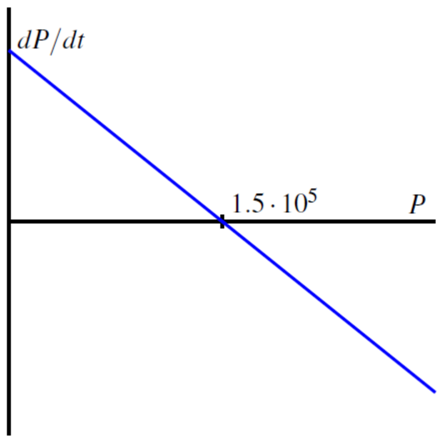
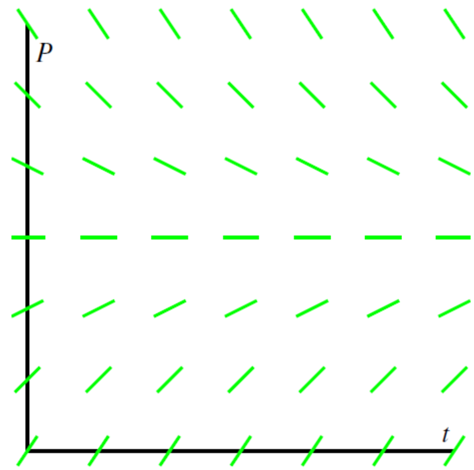
[1] and you assume that each cubic meter of water that flows out carries with it the plastic pollution it contains

Since this is an autonomous differential equation, you can sketch $\frac{dP}{dt}$ as a function of P and then construct a slope field, as shown below.





Plots of $\frac{dP}{dt}$ vs. P and the slope field for the differential equation d P =

These plots both show that $P = 1.5 \cdot 10^5$ is a stable equilibrium. Therefore, you should expect that the amount of pollution in Lake Michigan will stabilize near $1.5 \cdot 10^5$ cubic meters of pollution.

Next, assuming there is initially no pollution in the lake, you will solve the initial value problem

$$\frac{dP}{dt} = \frac{1}{100}(1.5 \cdot 10^5 - P), P(0) = 0.$$

Separating variables, you find that

$$\frac{1}{1.5 \cdot 10^5} \frac{dP}{dt} = \frac{1}{100}.$$

Integrating with respect to t, you have

$$\int \frac{1}{1.5 \cdot 10^5 - P} \frac{dP}{dt} dt = \int \frac{1}{100} dt$$

and thus changing variables on the left and antidifferentiating on both sides, you find that

$$\int \frac{dP}{1.5 \cdot 10^5 - P} = \int \frac{1}{100} dt$$

$$-\ln|1.5 \cdot 10^5 - P| = \frac{1}{100} t + C$$

Finally, multiplying both sides by -1 and using the definition of the logarithm, you find that

$$1.5 \cdot 10^5 - P = Ce^{-t/100} \quad \text{(Equation 2.1)}$$

This is a good time to determine the constant C. Since $P = 0$ when $t = 0$, you have

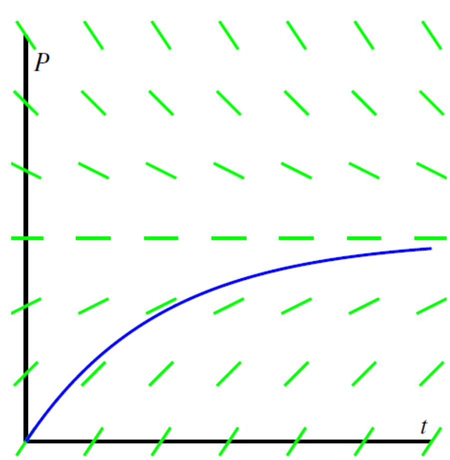
$$1.5 \cdot 10^5 - 0 = Ce^0 = C.$$

In other words, $C = 1.5 \cdot 105$.

Using this value of C in Equation (7.1) and solving for P, you arrive at the solution

$$P(t) = 1.5 \cdot 10^5 \left(1 - e^{-t/100}\right).$$

Superimposing the graph of P on the slope field you used earlier, you see, as shown in Figure 7.4 You see that, as expected, the amount of plastic pollution stabilizes around $1.5 \cdot 10^5$cubic meters.



There are many important lessons to learn from Example 2. Foremost is how you can develop a differential equation by thinking about the "total rate = rate in - rate out" model. In addition, you note how you can bring together all your available understanding (plotting $\frac{dP}{dt}$ vs. P, creating a slope field, solving the differential equation) to see how the differential equation describes the behavior of a changing quantity.

Of course, you can also explore what happens when certain aspects of the problem change. For instance, let's suppose you are at a time when the plastic pollution entering Lake Michigan has stabilized at $1.5 \cdot 10^5$ cubic meters, and that new legislation is passed to prevent this type of pollution entering the lake. So, there is no longer any inflow of plastic pollution to the lake. How does the amount of plastic pollution in Lake Michigan now change? For example, how long does it take for the amount of plastic pollution in the lake to halve?

Restarting the problem at time $t = 0$, you now have the modified initial value problem

$$\frac{dP}{dt} = -\frac{1}{100}P, P(0) = 1.5 \cdot 10^5.$$

It is a straightforward and familiar exercise to find that the solution to this equation is $P(t) = 1.5 \cdot 10^5 e^{-t/100}$. The time that it takes for half of the pollution to flow out of the lake is given by T where $P(t) = 0.75 \cdot 10^5$. Thus, you must solve the equation

$$0.75 \cdot 10^5 = 1.5 \cdot 10^5 e^{-T/100},$$

or

$$\frac{1}{2} = e^{-T/100}$$

It follows that

$$T = -100 \ln\left(\frac{1}{2}\right) \approx 69.3 \text{ years.}$$

In the upcoming investigations, you will explore some other natural settings in which differential equation model changing quantities.

Investigation 3

Suppose you have a bank account that grows by 5% every year. Let $A(t)$ be the amount of money in the account in year t.

3a) What is the rate of change of A with respect to t?

b. Suppose that you are also withdrawing $10,000 per year. Write a differential equation that expresses the total rate of change of A.

c. Sketch a slope field for this differential equation, find any equilibrium solutions, and identify them as either stable or unstable. Write a sentence or two that describes the significance of the stability of the equilibrium solution.

d. Suppose that you initially deposit $100,000 into the account. How long does it take for you to deplete the account?

e. What is the smallest amount of money you would need to have in the account to guarantee that you never deplete the money in the account?

f. If your initial deposit is $300,000, how much could you withdraw every year without depleting the account?

Investigation 4

A dose of morphine is absorbed from the bloodstream of a patient at a rate proportional to the amount in the bloodstream.

4a) Write a differential equation for $M(t)$, the amount of morphine in the patient's bloodstream, using k as the constant proportionality.

b. Assuming that the initial dose of morphine is $M(0)$, solve the initial value problem to find $M(t)$. Use the fact that the half-life for the absorption of morphine is two hours to find the constant k.

c. Suppose that a patient is given morphine intravenously at the rate of 3 milligrams per hour. Write a differential equation that combines the intravenous administration of morphine with the body's natural absorption.

d. Find any equilibrium solutions and determine their stability.

e. Assuming there is initially no morphine in the patient's bloodstream, solve the initial value problem to determine $M(t)$. What happens to $M(t)$ after a very long time?

f. To what rate should a doctor reduce the intravenous rate so that there is eventually 7 milligrams of morphine in the patient's bloodstream?

III. Exercises

1. Congratulations, you just won the lottery! In one option presented to you, you will be paid one million dollars a year for the next 25 years. You can deposit this money in an account that will earn 5% each year.

a. Set up a differential equation that describes the rate of change in the amount of money in the account. Two factors cause the amount to grow—first, you are depositing one millon dollars per year and second, you are earning 5% interest.

b. If there is no amount of money in the account when you open it, how much money will you have in the account after 25 years?

c. The second option presented to you is to take a lump sum of 10 million dollars, which you will deposit into a similar account. How much money will you have in that account after 25 years?

d. Do you prefer the first or second option? Explain your thinking.

e. At what time does the amount of money in the account under the first option overtake the amount of money in the account under the second option?

2. When a skydiver jumps from a plane, gravity causes her downward velocity to increase at the rate of $g \approx 9.8$ meters per second squared. At the same time, wind resistance causes her velocity to decrease at a rate proportional to the velocity.

a. Using k to represent the constant of proportionality, write a differential equation that describes the rate of change of the skydiver's velocity.

b. Find any equilibrium solutions and decide whether they are stable or unstable. Your result should depend on k.

c. Suppose that the initial velocity is zero. Find the velocity $v(t)$.

d. A typical terminal velocity for a skydiver falling face down is 54 meters per second. What is the value of k for this skydiver?

e. How long does it take to reach 50% of the terminal velocity?

3. During the first few years of life, the rate at which a baby gains weight is proportional to the reciprocal of its weight.

3a. Express this fact as a differential equation.

b. Suppose that a baby weighs 8 pounds at birth and 9 pounds one month later. How much will he weigh at one year?

c. Do you think this is a realistic model for a long time?

4. Suppose that you have a water tank that holds 100 gallons of water. A briny solution, which contains 20 grams of salt per gallon, enters the tank at the rate of 3 gallons per minute.

At the same time, the solution is well mixed, and water is pumped out of the tank at the rate of 3 gallons per minute.

4a. Since 3 gallons enters the tank every minute and 3 gallons leaves every minute, what can you conclude about the volume of water in the tank.

b. How many grams of salt enters the tank every minute?

c. Suppose that $S(t)$ denotes the number of grams of salt in the tank in minute *t*. How many grams are there in each gallon in minute *t*?

d. Since water leaves the tank at 3 gallons per minute, how many grams of salt leave the tank each minute?

e. Write a differential equation that expresses the total rate of change of *S*.

f. Identify any equilibrium solutions and determine whether they are stable or unstable.

g. Suppose that there is initially no salt in the tank. Find the amount of salt $S(t)$

in minute *t*.

h. What happens to $S(t)$ after a very long time? Explain how you could have predicted this only knowing how much salt there is in each gallon of the briny solution that enters the tank.