3.4D I'm Waking Up to Ash and Dust…

Separable Differential Equations

In the last two lessons, you have seen several ways to approximate the solution to an initial value problem. Given the frequency with which differential equations arise in the real world, it would help to have techniques for finding explicit algebraic solutions of certain initial value problems. In this lesson, you will focus on a particular class of differential equations (called separable) and develop a method for finding algebraic formulas for solutions to these equations.

A ***separable differential equation*** is a differential equation whose algebraic structure permits the variables present to be separated. For instance, consider the equation

$$\frac{dy}{dt} = ty$$

Separating (or isolating) the variables t and y so that all occurrences of t appear on the right-hand side, and all occurrences of y appears on the left and multiply $\frac{dy}{dt}$. You may do this in the preceding differential equation by dividing both sides by y:

$$\frac{1}{y}\frac{dy}{dt} = t$$

Note: when you attempt to separate the variables in a differential equation, you require that the left-hand side be a product in which the derivative $\frac{dy}{dt}$ is one term.

Not every differential equation is separable. For example, if you consider the equation

$$\frac{dy}{dt} = t - y$$

it may seem natural to separate it by writing

$$y + \frac{dy}{dt} = t$$

As you will see, this will not be helpful since the left-hand side is not a product of a function of y with $\frac{dy}{dt}$.

Investigation 1: In this preview activity, you explore whether certain differential equations are separable or not, and then revisit some key ideas from earlier work in integral calculus.

1a) Which of the following differential equations are separable? If the equation is separable, write the equation in the revised form $g(y)\frac{dy}{dt} = h(t)$.

i. $\frac{dy}{dt} = -3y$

ii. $\frac{dy}{dt} = ty - y$

iii. $\frac{dy}{dt} = t + 1$

iv. $\frac{dy}{dt} = t^2 - y^2$

b) Explain why any autonomous differential equation is guaranteed to be separable.

c) Why do you include the term "$+C$" in the expression

$$\int x\,dx = \frac{x^2}{2} + C$$

d) Suppose you know that a certain function f satisfies the equation

$$\int f'(x)\,dx = \int x\,dx$$

What can you conclude about f?

II. Solving separable differential equations

Before you examine a general approach to solving a separable differential equation, it is instructive to consider an example.

Example 1. Find all functions y that are solutions to the differential equation

$$\frac{dy}{dt} = \frac{t}{y^2}$$

Solution. Begin by separating the variables and writing

$$y^2 \frac{dy}{dt} = t$$

Integrating both sides of the equation with respect to the independent variable t shows that

$$\int y^2 \frac{dy}{dt} dt = \int t\, dt$$

Next, you notice that the left-hand side allows you to change the variable of antidifferentiation[1] from t to y. In particular, $dy = \frac{dy}{dt} dt$, so you now have

$$\int y^2\, dy = \int t\, dt$$

This most recent equation says that two families of antiderivatives are equal to one another. Therefore, when you find representative antiderivatives of both sides, you know they must differ by arbitrary constant C. Antidifferentiating and including the integration constant C on the right, you find that

$$\frac{y^3}{3} + C_1 = \frac{t^2}{2} + C_2$$

Notice the use of subscripts to distinguish between the arbitrary constant on each side of the equation; since $\frac{y^3}{3}$ and $\frac{t^2}{2}$ are in the same family of antiderivatives and must therefore differ by a single constant. The constants can be combined:

$$\frac{y^3}{3} = \frac{t^2}{2} + C_3$$

[1] This explains the requirement that the left-hand side be written as a product in which $\frac{dy}{dt}$ is one of the terms.

Finally, you may now solve the last equation above for y as a function of t, which gives

$$y(t) = \sqrt[3]{\frac{3}{2}t^2 + 3C_3}$$

Of course, the term$3C_3$ on the right-hand side represents 3 times an unknown constant. It is, therefore, still an unknown constant, which you will rewrite as C. You thus conclude that the function

$$y(t) = \sqrt[3]{\frac{3}{2}t^2 + C}$$

is a solution to the original differential equation for any value of C.

Notice that because this solution depends on the arbitrary constant C, you have found an infinite family of solutions. This makes sense because you expect to find a unique solution that corresponds to a given initial value.

For example, if you want to solve the initial value problem

$$\frac{dy}{dt} = \frac{t}{y^2}, \qquad y(0) = 2,$$

you know that the solution has the form $y(t) = \sqrt[3]{\frac{3}{2}t^2 + C}$ for some constant C. You

therefore must find the appropriate value for C that gives the initial value $y(0) = 2$. Hence,

$$2 = y(0) = \sqrt[3]{\frac{3}{2}0^2 + C} = \sqrt[3]{C}$$

which shows that $C = 2^3 = 8$. The solution to the initial value problem is then

$$y(t) = \sqrt[3]{\frac{3}{2}t^2 + 8}$$

The strategy of Example 1 may be applied to any differential equation of the form $\frac{dy}{dt} = g(y) \cdot h(t)$, and any differential equation of this form is said to be separable. You work to solve a separable differential equation by writing

$$\frac{1}{g(y)}\frac{dy}{dt} = h(t)$$

and integrating both sides with respect to *t*. After integrating, you then solve algebraically for *y* in order to write *y* as a function of *t*.

Consider one more example before doing further exploration in some activities.

Example 2. Solve the differential equation

$$\frac{dy}{dt} = 3y$$

Solution. Following the same strategy as in Example 1, you have

$$\frac{1}{y}\frac{dy}{dt} = 3$$

Integrating both sides with respect to *t*,

$$\int \frac{1}{y}\frac{dy}{dt}\,dt = \int 3\,dt,$$

and thus

$$\int \frac{1}{y}\,dy = \int 3\,dt.$$

Antidifferentiating and including the integration constant, you find that

$$\ln |y| = 3t + C_1$$

Finally, you need to solve for y. Here, one point deserves careful attention. By the definition of the natural logarithm function, it follows that

$$|y| = e^{3t+C_1} \cdot e^{C_2}$$

Since C_2 is an unknown constant, $\cdot\, e^{C_2}$ is as well, though you do know that it is positive (because e^x is positive for any x). When you remove the absolute value in order to solve for y, however, this constant may be either positive or negative. You will denote this updated constant (that accounts for a possible + or -) by C to obtain

$$y(t) = Ce^{3t}$$

There is one more slightly technical point to make. Notice that $y = 0$ is an equilibrium solution to this differential equation. In solving the equation above, you begin by dividing both sides by y, which is not allowed if $y = 0$. To be perfectly careful, therefore, you will typically consider the equilibrium solutions separately. In this case, notice that the final form of your solution captures the equilibrium solution by allowing $C = 0$.

Investigation 2: Suppose that the population of a town is growing continuously at an annual rate of 3% per year.

2a) Let $P(t)$ be the population of the town in year t. Write a differential equation that describes the annual growth rate.

b) Find the solutions of this differential equation.

c) If you know that the town's population in year 0 is 10,000, find the population $P(t)$.

d) How long does it take for the population to double? This time is called the ***doubling time***.

e) Working more generally, find the doubling time if the annual growth rate is k times the population.

Investigation 3: Suppose that a cup of coffee is initially at a temperature of 105°F and is placed in a 75°F room. Newton's law of cooling says that

$$\frac{dT}{dt} = -k(T - 75),$$

where k is a constant of proportionality.

a) Suppose you measure that the coffee is cooling at one degree per minute at the time the coffee is brought into the room. Use the differential equation to determine the value of the constant k.

b) Find all the solutions of this differential equation.

c) What happens to all the solutions as $t \to \infty$? Explain how this agrees with your intuition.

d) What is the temperature of the cup of coffee after 20 minutes?

e) How long does it take for the coffee to cool to 80°F?

Investigation 4: Solve each of the following differential equations or initial value problems.

a) $\frac{dy}{dt} - (2 - t)y = 2 - t$

b) $\frac{1}{t}\frac{dy}{dt} = e^{t^2 - 2y}$

(c) $y' = 2y + 2, \qquad y(0) = 2$

(d) $y' = 2y^2, \qquad y(-1) = 2$

(e) $\frac{dy}{dt} = \frac{-2ty}{t^2+1}, \qquad y(0) = 4$

III. Exercises

1. The mass of a radioactive sample decays at a rate that is proportional to its mass.

a) Express this fact as a differential equation for the mass $M(t)$ using k for the constant of proportionality.

b) If the initial mass is M_0, find an expression for the mass $M(t)$.

c) The half-life of the sample is the amount of time required for half of the mass to decay. Knowing that the half-life of Carbon-14 is 5730 years, find the value of k for a sample of Carbon-14.

d) How long does it take for a sample of Carbon-14 to be reduced to one-quarter its original mass?

e) Carbon-14 naturally occurs in your environment; any living organism takes in Carbon-14 when it eats and breathes. Upon dying, however, the organism no longer takes in Carbon-14.

Suppose that you find remnants of a pre-historic firepit. By analyzing the charred wood in the pit, you determine that the amount of Carbon-14 is only 30% of the amount in living trees. Estimate the age of the firepit.[2]

2. Consider the initial value problem

$$\frac{dT}{dt} = -\frac{t}{y}, \quad y(0) = 8$$

a) Find the solution of the initial value problem and sketch its graph.

b) For what values of t is the solution defined?

c) What is the value of y at the last time that the solution is defined?

d) By looking at the differential equation, explain why you should not expect to find solutions with the value of y you noted in (c).

[2] This approach is the basic idea behind radiocarbon dating.

3. Suppose that a cylindrical water tank with a hole in the bottom is filled with water. The water, of course, will leak out and the height of the water will decrease. Let $h(t)$ denote the height of the water. A physical principle called *Torricelli's Law* implies that the height decreases at a rate proportional to the square root of the height.

a) Express this fact using k as the constant of proportionality.

b) Suppose you have two tanks, one with $k = 1$ and another with $k = 10$. What physical differences would you expect to find?

c) Suppose you have a tank for which the height decreases at 20 inches per minute when the water is filled to a depth of 100 inches. Find the value of k.

d) Solve the initial value problem for the tank in part (c), and graph the solution you determine.

e) How long does it take for the water to run out of the tank?

f) Is the solution that you found valid for all time t? If so, explain how you know this. If not, explain why not.

4. The Gompertz equation is a model that is used to describe the growth of certain populations. Suppose that $P(t)$ is the population of some organism and that

$$\frac{dP}{dt} = -P \ln\left(\frac{P}{3}\right) = -P\ (\ln P - \ln 3)$$

a) Sketch a slope field for $P(t)$ over the range $0 \le P \le 6$.

b) Identify any equilibrium solutions and determine whether they are stable or unstable.

c) Find the population $P(t)$ assuming that $P(0) = 1$ and sketch its graph. What happens to $P(t)$ after a very long time?

(d) Find the population $P(t)$ assuming that $P(0) = 6$ and sketch its graph. What happens to P(t) after a very long time?

(e) Verify that the long-term behavior of your solutions agrees with what you predicted by looking at the slope field.

I. IV. Practice – Khan Academy

1. Complete the following online practice exercise in the *Differential Equations* unit of Khan Academy's AP Calculus AB course: https://www.khanacademy.org/math/ap-calculus-ab/diff-equations-ab/separable-eq-ab/e/separable-differential-equations