3.4b Qualitative Behavior in Differential Equations

Slope Fields, Equilibrium Solutions and more…

In earlier work, you have used the tangent line to the graph of a function f at a point a to approximate the values of f near a. The usefulness of this approximation is that you need to know very little about the function; armed with only the value $f(a)$ and the derivative $f'(a)$, you may find the equation of the tangent line and the approximation

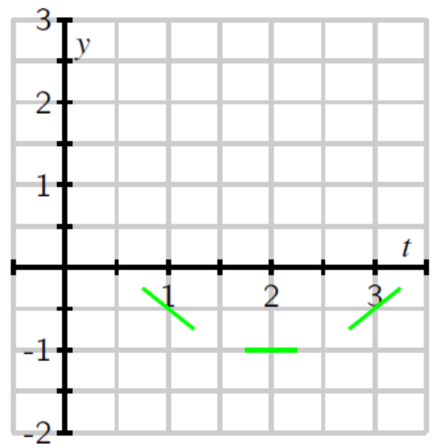
$$f(x) \approx f(a) + f'(a)(x - a).$$

Remember that a first-order differential equation gives you information about the derivative of an unknown function. Since the derivative at a point tells you the slope of the tangent line at this point, a differential equation gives you crucial information about the tangent lines to the graph of a solution. You will use this information about the tangent lines to create a ***slope field*** for the differential equation, which enables you to sketch solutions to initial value problems. Your aim will be to understand the solutions qualitatively. That is, you would like to understand the basic nature of solutions, such as their long-range behavior, without precisely determining the value of a solution at a particular point.

Investigation 1: Consider the initial value problem

$$\frac{dy}{dt} = t - 2, \quad y(0) = 1$$

a) Use the differential equation to find the slope of the tangent line to the solution $y(t)$ at $t = 0$. Then use the initial value to find the equation of the tangent line at $t = 0$. Sketch this tangent line over the interval $-0.25 \leq t \leq 0.25$ on the axes provided.



b) Also shown in the given figure are the tangent lines to the solution $y(t)$ at the points $t = 1$, 2, and 3 (you will see how to find these later). Use the graph to measure the slope of each tangent line and verify that each agrees with the value specified by the differential equation.

c) Using these tangent lines as a guide, sketch a graph of the solution $y(t)$ over the interval $0 \le t \le 3$ so that the lines are tangent to the graph of $y(t)$.

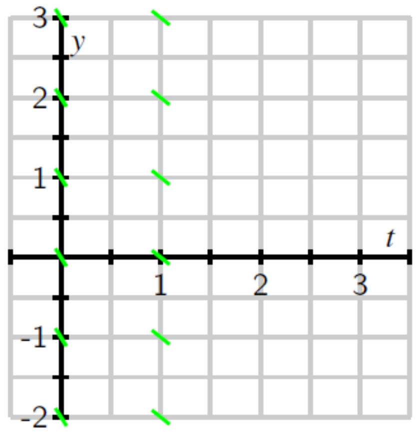
d) Use the Fundamental Theorem of Calculus to find $y(t)$, the solution to this initial value problem.

e) Graph the solution you found in (d) on the axes provided, and compare it to the sketch you made using the tangent lines.

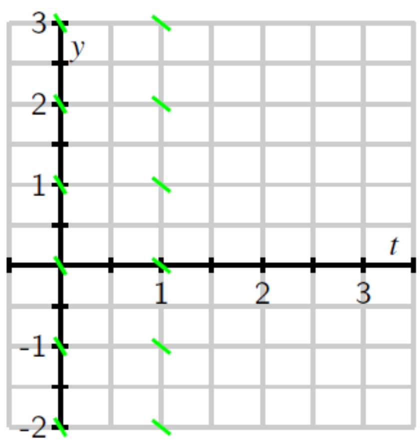
II. Slope fields

Investigation 1 shows that you may sketch the solution to an initial value problem if you know an appropriate collection of tangent lines. Because you may use a given differential equation to determine the slope of the tangent line at any point of interest, by plotting a useful collection of these, you can get an accurate sense of how certain solution curves must behave.

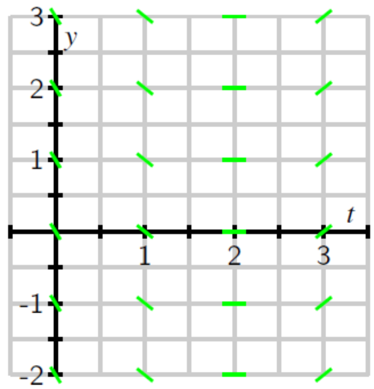
Let's continue looking at the differential equation $\frac{dy}{dt} = t - 2$. If t = 0, this equation says that $\frac{dy}{dt} = 0 - 2 = -2$. Notice that this value holds regardless of the value of y. You will therefore sketch tangent lines for several values of y and $t = 0$ with a slope of -2.



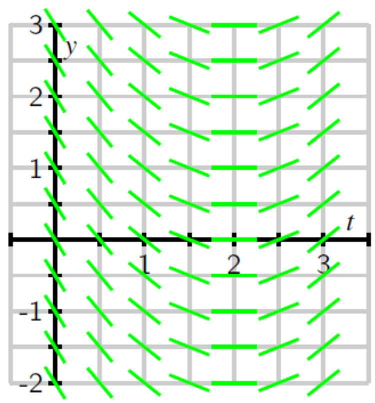
Let's continue in the same way: if $t = 1$, the differential equation tells us that $\frac{dy}{dt} = 1 - 2 = -1$., and this holds regardless of the value of y. You now sketch tangent lines for several values of y and $t = 1$ with a slope of -1.



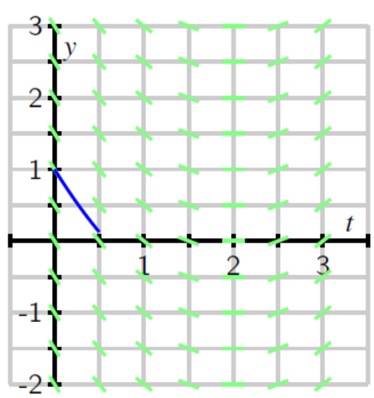
Similarly, you see that when $t = 2$, $\frac{dy}{dt} = 0$ and when $t = 3$, $\frac{dy}{dt} = 1$. You may therefore add to your growing collection of tangent line plots to achieve the next figure.



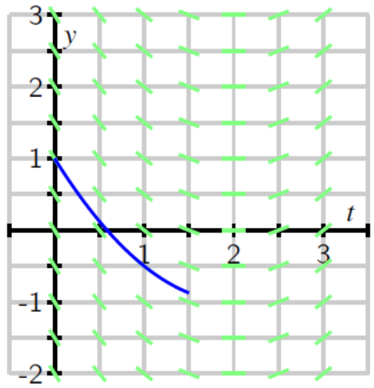
In this figure, you may see the solutions to the differential equation emerge. However, for the sake of clarity, you will add more tangent lines to provide the more complete picture shown below.

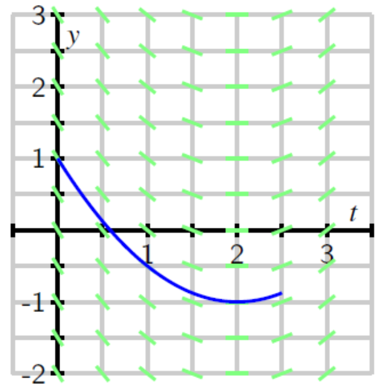


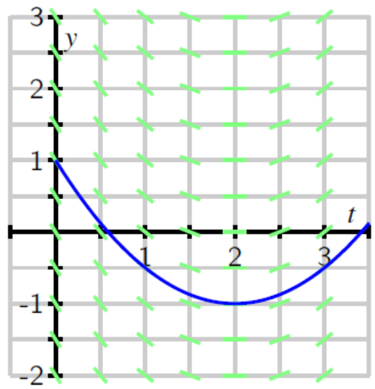
This most recent figure, which is called a slope field for the differential equation, allows you to sketch solutions just as you did in the preview activity. Here, you will begin with the initial value $y(0) = 1$ and start sketching the solution by following the tangent line, as shown in the next figure.



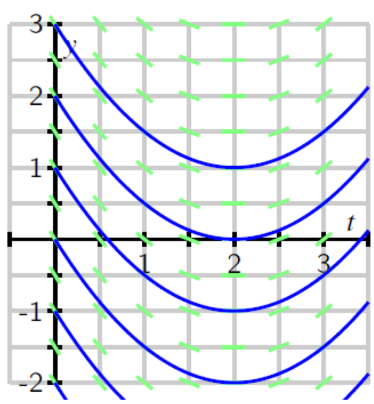
You then continue using this principle: whenever the solution passes through a point at which a tangent line is drawn, that line is tangent to the solution. Doing so leads you to the following sequence of images.







In fact, you may draw solutions for any possible initial value, and doing this for several different initial values for $y(0)$ results in the graphs shown next.

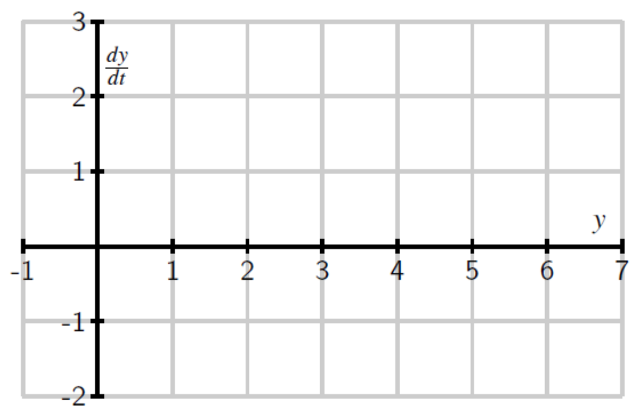


Just as you have done for the most recent example with $\frac{dy}{dt} = t - 2$, you can construct a slope field for any differential equation of interest. The slope field provides you with visual information about how you expect solutions to the differential equation to behave.

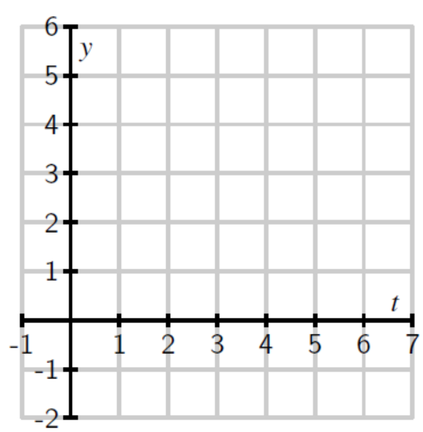
Investigation 2: Consider the autonomous differential equation

$$\frac{dy}{dt} = -\frac{1}{2}(y - 4).$$

a) Make a plot of $\frac{dy}{dt}$ versus y on the axes provided. Looking at the graph, for what values of y does $\frac{dy}{dt}$ increase and for what values of y does $\frac{dy}{dt}$ decrease?



b) Next, sketch the slope field for this differential equation on the axes provided.



c) Use your work in (b) to sketch the solutions that satisfy $y(0) = 0, y(0) = 2, y(0) = 4$ and $y(0) = 6$.

d) Verify that $y(t) = 4 + 2e^{-t/2}$ is a solution to the given differential equation with the initial value $y(0) = 6$. Compare its graph to the one you sketched in (c).

e) What is special about the solution where $y(0) = 4$?

III. Equilibrium solutions and stability

As your work in Investigation 2 demonstrates, first-order autonomous solutions may have solutions that are constant. In fact, these are quite easy to detect by inspecting the differential equation $\frac{dy}{dt} = f(y)$: constant solutions necessarily have a zero derivative so $\frac{dy}{dt} = 0 = f(y)$.

For example, in Investigation 2, you considered the equation

$$\frac{dy}{dt} = -\frac{1}{2}(y - 4).$$

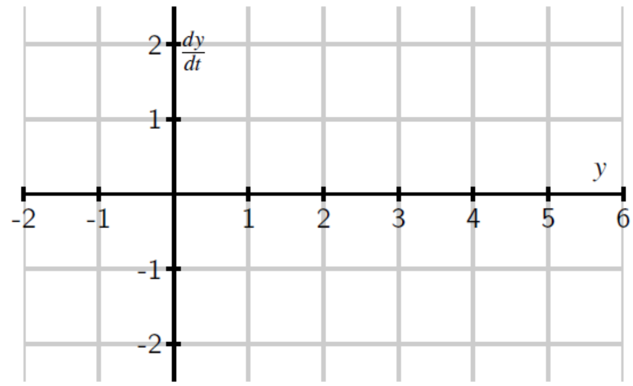
Constant solutions are found by setting $f(y) = -\frac{1}{2}(y - 4) = 0$, which you immediately see implies that $y = 4$.

Values of y for which $f(y) = 0$ in an autonomous differential equation $\frac{dy}{dt} = f(y)$ are usually called or ***equilibrium solutions*** of the differential equation.

Investigation 3: Consider the autonomous differential equation

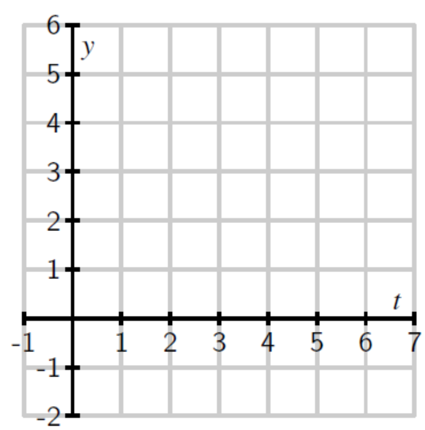
$$\frac{dy}{dt} = -\frac{1}{2}y(y - 4).$$

a) Make a plot of $\frac{dy}{dt}$ versus y. Looking at the graph, for what values of y does $\frac{dy}{dt}$ increase and for what values of y does $\frac{dy}{dt}$ decrease?



b) Identify any equilibrium solutions of the given differential equation.

c) Now sketch the slope field for the given differential equation.



d) Sketch the solutions to the given differential equation that correspond to initial values $y(0) = \text{-}1, 0, 1, \ldots, 5$.

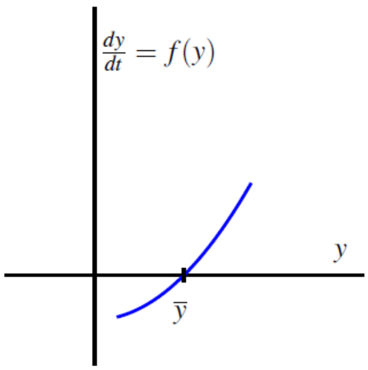
e) An equilibrium solution y is called ***stable*** if nearby solutions converge to $\bar{y}$. This means that if the initial condition varies slightly from y, then $\lim_{t \to \infty} y(t) = \bar{y}$.

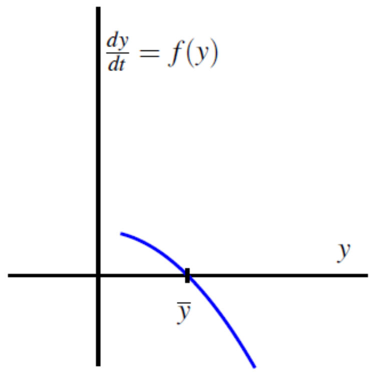
Conversely, an equilibrium solution y is called ***unstable*** if nearby solutions are pushed away from y.

Using your work above, classify the equilibrium solutions you found in (b) as either stable or unstable.

f) Suppose that $y(t)$ describes the population of a species of living organisms and that the initial value y_0 is positive. What can you say about the eventual fate of this population?

g) Remember that an equilibrium solution y satisfies $f(y) = 0$. If you graph $\frac{dy}{dt} = f(y)$ as a function of y, for which of the following differential equations is y a stable equilibrium and for which is y unstable? Why?





IV.Exercises

1. Consider the differential equation

$$\frac{dy}{dt} = t - y.$$

a) Sketch a slope field on the plot below:

b) Sketch the solutions whose initial values are $y(0) =$ -4, -3, . . . , 4.

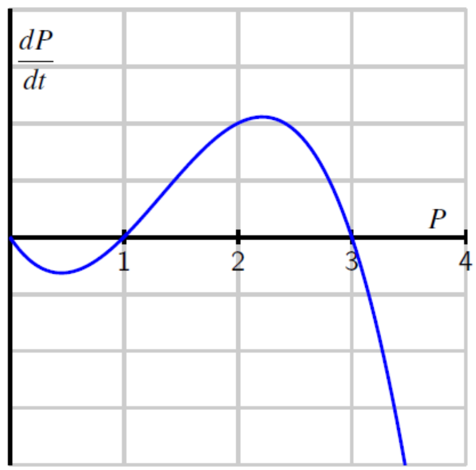
c) What do your sketches suggest is the solution whose initial value is $y(0) = 1$? Verify that this is indeed the solution to this initial value problem.

d) By considering the differential equation and the graphs you have sketched, what is the relationship between t and y at a point where a solution has a local minimum?

2. The population of a particular species is described by the function $P(t)$, where P is expressed in millions. Suppose further that the population's rate of change is governed by the differential equation

$$\frac{dP}{dt} = f(p).$$

where $f(p)$ is the function graphed below.



a) Sketch a slope field for this differential equation. You do not have enough information to determine the actual slopes, but you should have enough information to determine where slopes are positive, negative, zero, large, or small, and hence determine the qualitative behavior of solutions.

b) Sketch some solutions to this differential equation when the initial population

$P(0) > 0$.

c) Identify any equilibrium solutions to the differential equation and classify them as stable or unstable.

d) If $P(0) > 1$, what is the eventual fate of the species?

e) if $P(0) < 1$, what is the eventual fate of the species?

f) Remember that you referred to this model for population growth as "growth with a threshold." Explain why this characterization makes sense by considering solutions whose initial value is close to 1.

3. The population of a species of fish in a lake is $P(t)$ where P is measured in thousands of fish and t is measured in months. The growth of the population is described by the differential equation

$$\frac{dP}{dt} = f(p) = P(6 - P).$$

a) Sketch a graph of $f(p) = P(6 - P)$ and use it to determine the equilibrium solutions and whether they are stable or unstable. Write a complete sentence that describes the long-term behavior of the fish population.

b) Suppose now that the owners of the lake allow fishers to remove 1000 fish from the lake every month (remember that $P(t)$ is measured in thousands of fish). Modify the differential equation to take this into account. Sketch the new graph of $\frac{dP}{dt}$ versus P. Determine the new equilibrium solutions and decide whether they are stable or unstable.

c) Given the situation in part (b), give a description of the long-term behavior of the fish population.

d) Suppose that fishermen remove h thousand fish per month. How is the differential equation modified?

e) What is the largest number of fish that can be removed per month without eliminating the fish population? If fish are removed at this maximum rate, what is the eventual population of fish?

4. Let (t) be the number of thousands of mice that live on a farm; assume time t is measured in years.2

a) The population of the mice grows at a yearly rate that is twenty times the number of mice. Express this as a differential equation.

b) At some point, the farmer brings C cats to the farm. The number of mice that the cats can eat in a year is

$$M(y) = C\frac{y}{2 + y}$$

thousand mice per year. Explain how this modifies the differential equation that you found in part a).

c) Sketch a graph of the function $M(y)$ for a single cat $C = 1$ and explain its features by looking, for instance, at the behavior of $M(y)$ when y is small and when y is large.

d) Suppose that $C = 1$. Find the equilibrium solutions and determine whether they are stable or unstable. Use this to explain the long-term behavior of the mice population depending on the initial population of the mice.

e) Suppose that $C = 60$. Find the equilibrium solutions and determine whether they are stable or unstable. Use this to explain the long-term behavior of the mice population depending on the initial population of the mice.

f) What is the smallest number of cats you would need to keep the mice population from growing arbitrarily large?

V. Practice – Khan Academy

1. Complete the two following online practice exercises in the *Differential Equations* unit (Continuity) of Khan Academy's AP Calculus AB course:
    a. https://www.khanacademy.org/math/ap-calculus-ab/diff-equations-ab/slope-fields-ab/e/slope-fields
    b. https://www.khanacademy.org/math/ap-calculus-ab/diff-equations-ab/slope-fields-ab/e/slope-fields-and-solutions