3.4a Changing Gears

Introduction to Differential Equations

In previous lessons, you learned that a function's derivative tells the rate at which the function is changing. More recently, the Fundamental Theorem of Calculus helped you to determine the total change of a function over an interval when we know the function's rate of change. In this lesson, you will investigate the concept of differential equations. Simply said, a differential equation is an equation that provides a description of a function's derivative, which means that it tells us the function's rate of change. Using this information, we would like to learn as much as possible about the function itself. For instance, we would ideally like to have an algebraic description of the function.

Investigation1: The position of a moving object is given by the function $s(t)$, where s is measured in feet and t in seconds. The velocity is $v(t) = 4t + 1$ feet per second.

a) How much does the position change over the time interval [0, 4]?

b) Does this give you enough information to determine $s(4)$, the position at time $t = 4$? If so, what is $s(4)$? If not, what additional information would you need to know to determine $s(4)$?

c) Suppose you are told that the object's initial position $s\ 0 = 7$. Determine $s\ (2)$, the object's position 2 seconds later.

d) If you are told instead that the object's initial position is $s(0) = 3$, what is $s(2)$?

e) If you only know the velocity $v(t) = 4t + 1$, is it possible that the object's position at all times is $s(t) = 2t^2 + t - 4$? Explain how you know.

f) Are there other possibilities for $s(t)$? If so, what are they?

g) If, in addition to knowing the velocity function is $v(t) = 4t + 1$, we know the initial position $s(0)$, how many possibilities are there for $s(t)$?

II. What is a differential equation?

Definition: A differential equation is an equation that describes the derivative, or derivatives, of a function that is unknown to us. For instance, the equation

$$\frac{dy}{dx} = x\ sin(x)$$

is a differential equation since it describes the derivative of a function y (x) that is unknown.

As many important examples of differential equations involve quantities that change in time, the independent variable will frequently be time t. For instance, in the preview activity, you considered the differential equation

$$\frac{dv}{dt} = 4t + 1$$

Knowing the velocity and the starting position of the object, you may find the position at any later time.

Because differential equations describe the derivative of a function, they give information about how that function changes. Your goal will be to take this information and use it to predict the value of the function in the future; in this way, differential equations provide us with something like a crystal ball.

Differential equations arise frequently in our everyday world. For instance, you may hear a bank advertising:

Your money will grow at a 3% annual interest rate with us.

This innocuous statement is really a differential equation. Let's translate: A(t) will be amount of money you have in your account at time t. On one hand, the rate at which your money grows is the derivative $\frac{da}{dt}$. On the other hand, you are told that this rate is 0.03 A. This leads to the differential equation

$$\frac{da}{dt} = 0.03A$$

This differential equation has a slightly different feel than the previous equation $\frac{dv}{dt} = 4t + 1$. In the earlier example, the rate of change depends only on the independent variable t, and we may find s(t) by integrating the velocity 4t + 1. In the banking example, however, the rate of change depends on the dependent variable A, so you'll need some new techniques in order to find A(t).

Investigation 2: Express the following statements as differential equations. In each case, you will need to introduce notation to describe the important quantities in the statement so be sure to clearly state what your notation means.

a) The population of a town grows continuously at an annual rate of 1.25%.

b) A radioactive sample loses 5.6% of its mass every day.

c) You have a bank account that continuously earns 4% interest every year. At the same time, you withdraw money continually from the account at the rate of $1000 per year.

d) A cup of hot chocolate is sitting in a 70°F room. The temperature of the hot chocolate cools continuously by 10% of the difference between the hot chocolate's temperature and the room temperature every minute.

e) A can of cold soda is sitting in a 70°F room. The temperature of the soda warms continuously at the rate of 10% of the difference between the soda's temperature and the room's temperature every minute.

III. Differential equations in the world around us

As noted, differential equations give a natural way to describe phenomena in the real world. For instance, physical principles are frequently expressed as a description of how a quantity changes. A good example is Newton's Second Law, an important physical principle that says:

The product of an object's mass and acceleration equals the force applied to it.

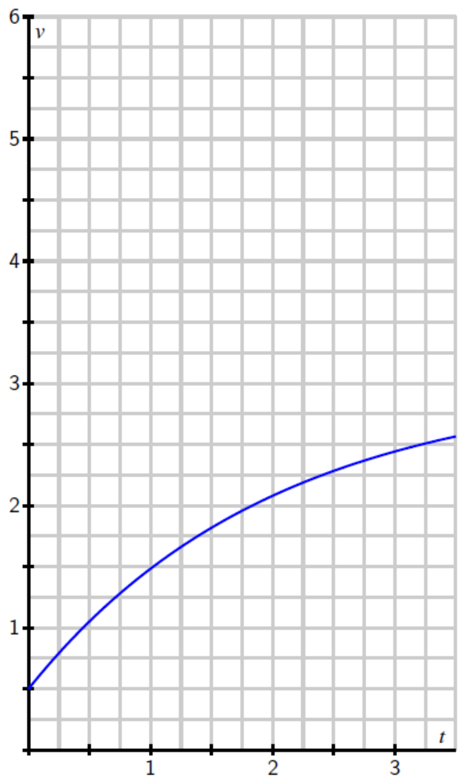
For instance, when gravity acts on an object near the earth's surface, it exerts a force equal to mg, the mass of the object times the gravitational constant g. We therefore have

$$\mathrm{ma} = \mathrm{mg}, \quad \text{or}$$

$$\frac{dv}{dt} = g,$$

where v is the velocity of the object, and $g = 9.8$ meters per second squared. Notice that this physical principle does not tell us what the object's velocity is, but rather how the object's velocity changes.

Investigation 3: Shown below are two graphs depicting the velocity of falling objects. On the left is the velocity of a skydiver, while on the right is the velocity of a meteorite entering the Earth's atmosphere.



Skydiver's velocity

Meteorite's velocity

a) Begin with the skydiver's velocity and use the given graph to measure the rate of change $\frac{dv}{dt}$ when the velocity is v = 0.5, 1.0, 1.5, 2.0, and 2.5. Plot your values on the graph below. You will want to think carefully about this: you are plotting the derivative $\frac{dv}{dt}$ as a function of velocity.

b) Now do the same thing with the meteorite's velocity: use the given graph to measure the rate of change $\frac{dv}{dt}$ when the velocity is v = 3.5, 4.0, 4.5, and 5.0. Plot your values on the graph above.

c) You should find that all your points lie on a line. Write the equation of this line being careful to use proper notation for the quantities on the horizontal and vertical axes.

d) The relationship you just found is a differential equation. Write a complete sentence that explains its meaning.

e) By looking at the differential equation, determine the values of the velocity for which the velocity increases.

f) By looking at the differential equation, determine the values of the velocity for which the velocity decreases.

g) By looking at the differential equation, determine the values of the velocity for which the velocity remains constant.

The point of this activity is to demonstrate how differential equations model processes in the real world. In this example, two factors are influencing the velocities: gravity and wind resistance. The differential equation describes how these factors influence the rate of change of the objects' velocities.

IV. Solving a differential equation

To solve a differential equation means to find a function that satisfies the description given.

For instance, the first differential equation we looked at is

$$\frac{dv}{dt} = 4t + 1$$

which describes an unknown function s(t). We may check that $s(t) = 2t^2 + t$ is a solution because it satisfies this description. Notice that $s(t) = 2t^2 + t + 4$ is also a solution.

If you have a candidate for a solution, it is straightforward to check whether it is a solution or not.

Consider the differential equation

$$\frac{dv}{dt} = 4t + 1$$

Let’s ask whether v (t) = 3 - 2$e^{-0.5t}$ is a solution1. Using this formula for v, observe first that

$$\frac{dv}{dt} = \frac{d}{dt}[3 - 2e^{-0.5t} = -2e^{-0.5t} \cdot (-0.5) = e^{-0.5t}$$

1.5 - 0.5v = 1.5 - 0.5o = 1.5 - 0.5(3 2$e^{-0.5t}$) = 1.5 - 1.5 + $e^{-0.5t}$ = $e^{-0.5t}$.

Since $\frac{dv}{dt}$ and 1.5 - 0.5v agree for all values of t when v = 3 2$e^{-0.5t}$, you have found a solution to the differential equation.

Investigation 3: Consider the differential equation $\frac{dv}{dt} = 1.5 - 0.5v$. Which of the following functions are solutions of this differential equation?

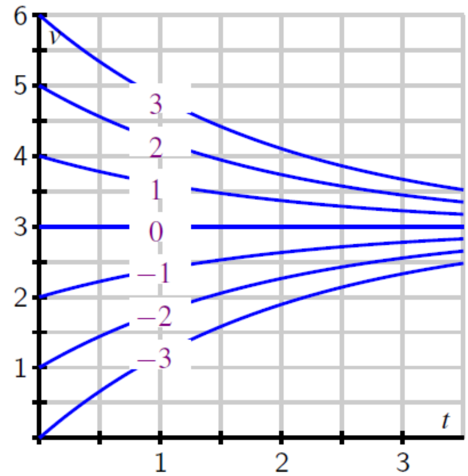
a) v(t) = 1.5t - 0.25t^2.

b) v(t) = 3 + 2$e^{-0.5t}$.

c) v(t) = 3.

d) v(t) = 3 + Ce?0.5t where C is any constant.

This activity shows us something interesting. Notice that the differential equation has infinitely many solutions, which are parametrized by the constant C in v(t) = 3 + C$e^{-0.5t}$. In the figure below, you see the graphs of these solutions for a few values of C, as labeled.



Notice that the value of C is connected to the initial value of the velocity v (0) , since v (0) = 3 + C. In other words, while the differential equation describes how the velocity changes as a function of the velocity itself, this is not enough information to determine the velocity uniquely: you also need to know the initial velocity. For this reason, differential equations will typically have infinitely many solutions, one corresponding to each initial value. We have seen this phenomenon before, such as when given the velocity of a moving object v (t) , you were not able to uniquely determine the object's position unless you also know its initial position.

If given a differential equation and an initial value for the unknown function, this is an initial value problem. For instance,

$$\frac{dv}{dt} = 1.5 - 0.5v, v(0) = 0.5$$

is an initial value problem. In this situation, we know the value of v at one time and we know how v is changing. Consequently, there should be exactly one function v that satisfies the initial value problem.

This demonstrates the following important general property of initial value problems.

We won't worry about what "well behaved" means—it is a technical condition that will be satisfied by all the differential equations we consider.

To close this section, we note that differential equations may be classified based on certain characteristics they may possess. Indeed, you may see many different types of differential equations in a later course in differential equations. For now, we would like to introduce a few terms that are used to describe differential equations.

A first-order differential equation is one in which only the first derivative of the function occurs. For this reason,

$$\frac{dv}{dt} = 1.5 - 0.5v$$

is a ***first-order*** equation, while

$$\frac{d^2y}{dt^2} = -10y$$

is a ***second-order*** equation. A differential equation is ***autonomous*** if the independent variable does not appear in the description of the derivative. For instance,

$$\frac{dv}{dt} = 1.5 - 0.5v$$

is autonomous because the description of the derivative dv/dt does not depend on time. The equation

$$\frac{dv}{dt} = 1.5t - 0.5y$$

however, is not autonomous.

V. Exercises

1. Suppose that T (t) represents the temperature of a cup of coffee set out in a room, where T is expressed in degrees Fahrenheit and t in minutes. A physical principle known as Newton's Law of Cooling tells us that

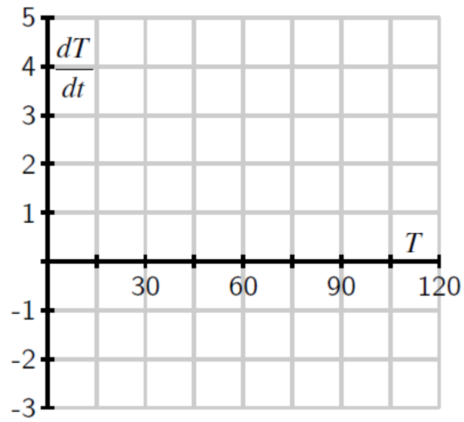
$$\frac{dT}{dt} = -\frac{1}{15}T + 5$$

a) Supposes that T (0) = 105. What does the differential equation give us for the value of $\frac{dT}{dt}T = 105$? Explain in a complete sentence the meaning of these two facts.

b) Is T increasing or decreasing at t = 0?

c) What is the approximate temperature at t = 1?

d) On the graph below, make a plot of dT /dt as a function of T .



e) For which values of T does T increase? For which values of T does T decrease?

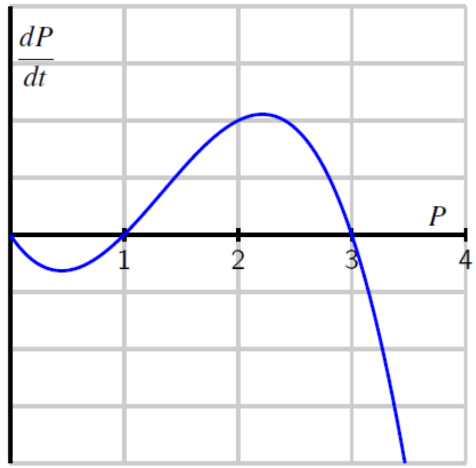
f) What do you think is the temperature of the room? Explain your thinking.

g) Verify that $T(t) = 75 + 30e^{-t/15}$ is the solution to the differential equation with initial value T (0) = 105. What happens to this solution after a long time?

2. Suppose that the population of a particular species is described by the function P t , where P is expressed in millions. Suppose further that the population's rate of change is governed by the differential equation

$$\frac{dP}{dt} = f(P)$$

where f (P) is the function graphed below.



a) For which values of the population P does the population increase?

b) For which values of the population P does the population decrease?

c) If $P(0) = 3$, how will the population change in time?

d) If the initial population satisfies $0 < P\,(0) < 1$, what will happen to the population after a very long time?

e) If the initial population satisfies $1 < P\,(0) < 3$, what will happen to the population after a very long time?

f) If the initial population satisfies $3 < P\,(0)$, what will happen to the population after a very long time?

g) This model for a population's growth is sometimes called "growth with a threshold." Explain why this is an appropriate name.

3. In this problem, you investigate further what it means for a function to be a solution to a given differential equation.

(a) Consider the differential equation

$$\frac{dy}{dt} = y - t$$

Determine whether the following functions are solutions to the given differential equation.

(i) $y(t) = t + 1 + 2e^t$

(ii) $y(t) = t + 1$

(iii) $y(t) = t + 2$

b) When you weigh bananas in a scale at the grocery store, the height h of the bananas is described by the differential equation

$$\frac{d^2h}{dt^2} = -kh$$

where k is the ***spring constant***, a constant that depends on the properties of the spring in the scale. After you put the bananas in the scale, you (cleverly) observe that the height of the bananas is given by $h\ t = 4\sin(3t)$. What is the value of the spring constant?

I. TI – Limits Basics – at end (opt.)

II. GeoGebra apps

1. https://ggbm.at/K8MnDYh4

III. IV. Practice – Khan Academy

1. Complete the six online practice exercises in the first unit (Continuity) of Khan Academy's AP Calculus AB course: https://www.khanacademy.org/math/ap-calculus-ab/continuity-ab?t=practice