3.3D Nothing Compares to *U*

Integration by Substitution and the Reverse Chain Rule

The goal of this lesson is to learn how to "undo" the process of differentiation to find an algebraic antiderivative for composite functions.

Investigation 1: In Big Idea 2, you learned the Chain Rule and how it can be applied to find the derivative of a composite function. If u is a differentiable function of x, and f is a differentiable function of $u(x)$, then

$$\frac{dy}{dx}\left[f\left(u(x)\right)\right] = f'(u(x)) \cdot u'(x).$$

In words, we say that the derivative of a composite function $c(x) = f\,(u(\,x))$, where f is considered the "outer" function and u the "inner" function, is "the derivative of the outer function, evaluated at the inner function, times the derivative of the inner function."

1a) For each of the following functions, use the Chain Rule to find the function's derivative. Be sure to label each derivative by name (e.g., the derivative of $g(x)$ should be labeled g'(x)).

i. $g(x) = e^{3x}$

ii. $h(x) = \sin(5x + 1)$

iii. $p(x) = \arctan(2x)$

iv. $q(x) = (2 - 7x)^4$

v. $r(x) = 3^{4-11x}$

b) For each of the following functions, use your work in (a) to help you determine the general antiderivative[1] of the function. Label each antiderivative by name (e.g., the antiderivative of m should be called M). In addition, check your work by computing the derivative of each proposed antiderivative.

i. $m(x) = \text{e}3x$

ii. $n(x) = \cos(5x + 1)$

iii. $s(x) = \dfrac{1}{1 + 4x^2}$

iv. $v(x) = (2 - 7x)^3$

v. $w(x) = 3^{4-11x}$

c) Based on your experience in parts (a) and (b), conjecture an antiderivative for each of the following functions. Test your conjectures by computing the derivative of each proposed antiderivative.

i. $a(x) = \cos(\pi x)$

ii. $b(x) = (4x + 7)^{11}$

iii. $s(x) = xe^{x^2}$

[1] Recall that the general antiderivative of a function includes "+C" to reflect the entire family of functions that share the same derivative.

II. Reversing the Chain Rule: First Steps

Investigation 1 led you to antidifferentiate a function of the form

$$h(x) = f(u(x)),$$

whenever f is a familiar function whose antiderivative is known and $u\ x$ is a linear function. For example, if we consider

$$h(x) = (5x - 3)^6,$$

in this context the outer function f is $f(u) = u^6$, while the inner function is $u(x) = 5x$. Since the antiderivative of f is $F(u) = \frac{1}{7}u^7 + C$, we see that the antiderivative of h is

$$H(x) = \frac{1}{7}(5x - 3)^7 \cdot \frac{1}{5} + C$$

$$= \frac{1}{35}(5x - 3)^7 + C.$$

The inclusion of the constant 1 is essential precisely because the derivative of the inner function is $u'(x) = 5$. Indeed, if you now compute $H'(x)$, you find by the Chain Rule (and Constant Multiple Rule) that

$$H'(x) = \frac{1}{35}(5x - 3)^6 \cdot 5 = (5x - 3)^6 = h(x)$$

and thus H is indeed the general antiderivative of h.

Hence, in the special case where the outer function is familiar and the inner function is linear, you can antidifferentiate composite functions according to the following rule.

> If $h(x) = f(ax+b)$ and F is a known algebraic antiderivative of f, then h is given the rule
>
> $$H(x) = \frac{1}{a}(ax + b) + C$$

When discussing antiderivatives, it is often useful to have shorthand notation that indicates the instruction to find an antiderivative. Thus, in a similar way to how the notation

$$\frac{d}{dx}[f(x)]$$

represents the derivative of $f(x)$ with respect to x, one uses the notation of the ***indefinite integral***,

$$\int f(x)dx$$

to represent the general antiderivative of f with respect to x. For instance, returning to the earlier example with $h(x) = (5x - 3)^6$ above, we can rephrase the relationship between h and its antiderivative H through the notation

$$\int (5x-3)^6 dx = \frac{1}{35}(5x-6)^7 + C$$

When we find an antiderivative, we will often say that we ***evaluate an indefinite integral***; said differently, the instruction to evaluate an indefinite integral means to find the general antiderivative. Just as the notation $\frac{dy}{dx}[\blacksquare]$ means "find the derivative with respect to x of ■," the notation $\int \blacksquare\, dx$ means "find a function of x whose derivative is ■."

Investigation 2: Evaluate each of the following indefinite integrals. Check each antiderivative that you find by differentiating.

2a) $\int \sin(x-3)dx$

b) $\int \sec^2(4x)dx$

c) $\int \frac{1}{11x-9}dx$

d) $\int \csc(2x+1)\cot(2x+1)dx$

e) $\int \frac{1}{\sqrt{1-16x^2}} dx$

f) $\int 5^{-x}\, dx$

III. Reversing the Chain Rule: *u*-substitution

Of course, a natural question arises from our recent work: what happens when the inner function is not a linear function? For example, can we find antiderivatives of such functions as

$$g(x) = xe^{x^2} \quad \text{and} \quad h(x) = e^{x^2}$$

It is important to explicitly remember that differentiation and antidifferentiation are essentially inverse processes; that they are not quite inverse processes is due to the $+C$ that arises when antidifferentiating. This close relationship enables you to take any known derivative rule and translate it to a corresponding rule for an indefinite integral. For example, since

$$\frac{d}{dx}[x^5] = 5x^4$$

you can equivalently write

$$\int 5x^4 = x^5 + C.$$

Recall that the Chain Rule states that

$$\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$$

Restating this relationship in terms of an indefinite integral,

$$\int f'\big(g(x)\big) \cdot g'(x) dx = f\big(g(x)\big) + C \qquad \textbf{Equation 1}$$

Equation 1 reveals that if we can take a given function and view its algebraic structure as $f'(g(x)) \cdot g'(x)$ for some appropriate choices of f and g, then we can antidifferentiate the function by reversing the Chain Rule. It is especially notable that both $g(x)$ and $g'(x)$ appear in the form of $f'(g(x)) \cdot g'(x)$; we will sometimes say that we seek to identify a function-derivative pair when trying to apply the rule in Equation 1.

In the situation where you can identify a function-derivative pair, simply introduce a new variable u to represent the function $g(x)$. Observing that with $u = g(x)$, it follows in Leibniz notation that $\frac{du}{dx} = g'(x)$, so that in terms of differentials[2], $du = g'(x)\ dx$. Now converting the indefinite integral of interest to a new one in terms of u, you get

$$\int f'(g(x)) \cdot g'(x) = \int f'(u)du.$$

Provided that f , is an elementary function whose antiderivative is known, you can now easily evaluate the indefinite integral in u, and then go on to determine the desired overall antiderivative of $f'(g(x)) \cdot g'(x)$. This process is called *u*-substitution. To see *u*-substitution at work, consider the following example.

Example 1: Evaluate the indefinite integral

$$\int \sin(7x^4 + 3) \cdot x^3 dx$$

and check the result by differentiating.

Solution. Notice that x^3 is almost the derivative of $7x^4 + 3$; the only issue is a missing constant.

Let u represent the inner function of the composite function $\sin 7x^4 + 3$, we have u $= 7x^4 + 3$, and thus $\frac{du}{dx} = 28x^3$. In differential notation, it follows that du $= 28x^3\ dx$, and thus $x^3\ dx = 1\ du$. The original indefinite integral may now be written

$$\int \sin(u) \cdot \frac{1}{28} du$$

$$\frac{1}{28} \int \sin(u) du$$

$$\frac{1}{28}(-\cos(u) + C$$

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[2] If you remember from the definition of the derivative that $\frac{du}{dx} \approx \frac{\Delta u}{\Delta x}$ and use the fact that $\frac{du}{dx} = g'(x)$, then you will see that $g'(x) \approx \frac{\Delta u}{\Delta x}$. Solving for u, $\Delta u = g'(x)\Delta x$. It is this last relationship that, when expressed in differential notation enables us to write $du = g'(x)\ dx$ in the change of variable formula.

Now substitute the original inner function in place of u and you get

$$\frac{1}{28}(-\cos(7x^4+3)+C$$

To check your work, observe by the Chain Rule that

$$\frac{d}{dx}[\frac{1}{28}(-\cos(7x^4+3)+C] = -\frac{1}{28}\cdot(-1)\sin(7x^4+3)\cdot 28x^3$$

$$= \sin(7x^4+3)\cdot x^3$$

which is indeed the original integrand.

Investigation 3:

3. Evaluate each of the following indefinite integrals by using these steps:

- Find two functions within the integrand that form (up to a possible missing constant) a function-derivative pair;
- Make a substitution and convert the integral to one involving *u* and *du*;
- Evaluate the new integral in *u*;
- Convert the resulting function of *u* back to a function of *x* by using your earlier substitution;
- Check your work by differentiating the function of *x*. You should come up with the integrand originally given.

3a) $\int \frac{x^2}{5x^3+1}\,dx$

b) $\int e^x \sin(e^x)\,dx$

c) $\int \frac{\cos(\sqrt{x})}{\sqrt{x}}\,dx$

IV. Evaluating Definite Integrals via *u*-substitution

You have just learned *u*-substitution to evaluate indefinite integrals of functions that can be written, up to a constant multiple, in the form $f'(g(x)) \cdot g'(x)$. This same technique can be used to evaluate definite integrals involving such functions, though you must be careful with the corresponding limits of integration. Consider, for instance, the definite integral

$$\int_2^5 xe^{x^2}\,dx$$

Whenever you write a definite integral, it is implicit that the limits of integration correspond to the variable of integration. To be more explicit, observe that

$$\int_2^5 xe^{x^2}\,dx = \int_{x=2}^{x=5} xe^{x^2}dx.$$

When you execute the *u*-substitution, change the variable of integration; it is essential to note that this also changes the limits of integration. For instance, with the substitution $u = x^2$ and $du = 2x\,dx$, it also follows that when $x = 2$, $u = 2^2 = 4$, and when $x = 5$, $u = 5^2 = 25$. Thus, under the change of variables of *u*-substitution, you now have

$$\int_{x=2}^{x=5} xe^{x^2}dx = \int_{u=4}^{u=25} e^u \cdot \frac{1}{2}du$$

$$= \frac{1}{2}e^u\Big|_{u=4}^{u=25}$$

$$\frac{1}{2}e^{25} - \frac{1}{2}e^4.$$

Investigation 4: Evaluate each of the following definite integrals exactly through an appropriate *u*-substitution.

4a) $\int_1^2 \frac{x}{1+4x^2}dx$

b) $\int_0^1 e^{-x}(2e^{-x}+3)^9$

c) $\int_{\frac{2}{\pi}}^{\frac{4}{\pi}} \frac{\cos(\frac{1}{x})}{x^2}\,dx$

V. Exercises

1. This problem centers on finding antiderivatives for the basic trigonometric functions other than $\sin(x)$ and $\cos(x)$.

a) Consider the indefinite integral $\int \tan(x)\,dx$. By rewriting the integrand as tan(x) $= \frac{\sin(x)}{\cos(x)}$ and identifying an appropriate function-derivative pair, make a *u*-substitution and hence evaluate $\int \tan(x)\,dx$.

b) In a similar way, evaluate $\int \cot(x)\,dx$.

c) Consider the indefinite integral

$$\int \frac{\sec^2(x) + \sec(x)\tan(x)}{\sec(x) + \tan(x)}$$

Evaluate this integral using the substitution $u = \sec(x) + \tan(x)$.

d) Simplify the integrand in (c) by factoring the numerator. What is a far simpler way to write the integrand?

e) Combine your work in (c) and (d) to determine $\int \sec(x)\,dx$.

f) Using (c)-(e) as a guide, evaluate $\int \csc(x)\,dx$.

2. Consider the indefinite integral $\int x\sqrt{x-1}\,dx$.

a) At first glance, this integrand may not seem suited to substitution due to the presence of *x* in separate locations in the integrand. Nonetheless, using the composite function $\sqrt{x-1}$ as a guide, let $u =$ x? 1. Determine expressions for both *x* and *dx* in terms of *u*.

b) Convert the given integral in *x* to a new integral in *u*.

c) Evaluate the integral in (b) by noting that $\sqrt{u} = u^{\frac{1}{2}}$ and observing that it is now possible to rewrite the integrand in *u* by expanding through multiplication.

d) Evaluate each of the integrals $\int x^2\sqrt{x-1}\,dx$? 1 dx and $\int x\sqrt{x^2-1}\,dx$. Write a paragraph to discuss the similarities among the three indefinite integrals in this problem and the role of substitution and algebraic rearrangement in each.

3. Consider the indefinite integral $\int \sin^2(x)dx$

a) Explain why the substitution $u = \sin(x)$will not work to help evaluate the given integral.

b) Recall the Fundamental Trigonometric Identity, which states that $\sin^2(x) + \cos^2(x) = 1$. By observing that $\sin^3 x = \sin(x)\sin^2(x)$, use the Fundamental Trigonometric Identity to rewrite the integrand as the product of $\sin(x)$ with another function.

c) Explain why the substitution $u = \cos x$ now provides a possible way to evaluate the integral in (b).

d) Use your work in (a)-(c) to evaluate the indefinite integral $\int \sin^3(x)dx$.

(e) Use a similar approach to evaluate $\int \cos^3(x)dx$.

4. For the town of Mathland, MI, residential power consumption has shown certain trends over recent years. Based on data reflecting average usage, engineers at the power company have modeled the town's rate of energy consumption by the function

$$r(t) = 4 + \sin(0.263t + 4.7) + \cos(0.526t + 9.4).$$

Here, t measures time in hours after midnight on a typical weekday, and r is the rate of consumption in megawatts at time t. Units are critical throughout this problem.

a) Sketch a carefully labeled graph of $r(t)$ on the interval $[0,24]$ and explain its meaning. Why is this a reasonable model of power consumption?

b) Without calculating its value, explain the meaning of $\int_0^{24} r(t)dt$. Include appropriate units on your answer.

c) Determine the exact amount of power Mathland consumes in a typical day.

d) What is Mathland's average rate of energy consumption in a given 24-hour period? What are the units on this quantity?

VI. Assessment – Khan Academy

1. Complete 3 out of 4 (the first two and the last) practice exercises in the Integration Techniques unit (Continuity) of Khan Academy's AP Calculus AB course: https://www.khanacademy.org/math/ap-calculus-ab/integration-techniques-ab?t=practice

2. Optional Practice 3 (Challenge)

5.4 Integration by Parts

Introduction

In Section 5.3, we learned the technique of *u*-substitution for evaluating indefinite integrals that involve certain composite functions. For example, the indefinite integral x3 sin x4 dx is perfectly suited to *u*-substitution, since not only is there a composite function present, but also the inner function's derivative (up to a constant) is multiplying the composite function. Through *u*-substitution, we learned a general situation where

recognizing the algebraic structure of a function can enable us to find its antiderivative.

It is natural to ask similar questions to those we considered in Section 5.3 about functions with a different elementary algebraic structure: those that are the product of basic functions. For instance, suppose we are interested in evaluating the indefinite integral

r x sin(x) dx.

Here, there is not a composite function present, but rather a product of the basic functions f x = x and g x = sin x . From our work in Section 2.3 with the Product Rule, we know that it is relatively complicated to compute the derivative of the product of two functions, so we should expect that antidifferentiating a product should be similarly involved. In addition, intuitively we expect that evaluating x sin x dx will involve somehow reversing the Product Rule.

To that end, in Preview Activity 5.4 we refresh our understanding of the Product Rule and then investigate some indefinite integrals that involve products of basic functions.

Preview Activity 5.4. In Section 2.3, we developed the Product Rule and studied how it is employed to differentiate a product of two functions. In particular, recall that if f and g are differentiable functions of x, then

dx [f (x) · g(x)] = f (x) · g,(x) + g(x) · f ,(x).

(a) For each of the following functions, use the Product Rule to find the function's derivative. Be sure to label each derivative by name (e.g., the derivative of g(x) should be labeled g,(x)).

i. g(x) = x sin(x)

ii. h(x) = xex

iii. p(x) = x ln(x)

iv. q(x) = x2 cos(x)

v. r(x) = ex sin(x)

(b) Use your work in (a) to help you evaluate the following indefinite integrals. Use differentiation to check your work.

i. r xex + ex dx

ii. r ex (sin(x) + cos(x)) dx

iii. r 2x cos(x) ? x2 sin(x) dx

iv. r x cos(x) + sin(x) dx

v. r 1 + ln(x) dx

(c) Observe that the examples in (b) work nicely because of the derivatives you were asked to calculate in (a). Each integrand in (b) is precisely the result of differentiating one of the products of basic functions found in (a). To see what happens when an integrand is still a product but not necessarily the result of differentiating an elementary product, we consider how to evaluate

r x cos(x) dx.

i. First, observe that

d

dx [x sin(x)] = x cos(x) + sin(x).

Integrating both sides indefinitely and using the fact that the integral of a sum is the sum of the integrals, we find that

r i d [x sin(x)]\ dx = r x cos(x) dx + r sin(x) dx.

In this last equation, evaluate the indefinite integral on the left side as well as the rightmost indefinite integral on the right.

ii. In the most recent equation from (i.), solve the equation for the expression

x cos(x) dx.

iii. For which product of basic functions have you now found the antiderivative?

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Reversing the Product Rule: Integration by Parts

Problem (c) in Preview Activity 5.4 provides a clue for how we develop the general technique known as Integration by Parts, which comes from reversing the Product

Rule. Recall that the Product Rule states that

d [f (x)g(x)] = f (x)g,(x) + g(x) f ,(x).

Integrating both sides of this equation indefinitely with respect to x, it follows that

r d [f (x)g(x)] dx = r f (x)g,(x) dx + r g(x) f ,(x) dx. (5.6)

On the left in Equation (5.6), we recognize that we have the indefinite integral of the derivative of a function which, up to an additional constant, is the original function itself.

Temporarily omitting the constant that may arise, we equivalently have

f (x)g(x) = r f (x)g,(x) dx + r g(x) f ,(x) dx. (5.7)

The most important thing to observe about Equation (5.7) is that it provides us with a choice of two integrals to evaluate. That is, in a situation where we can identify two functions f and g, if we can integrate f (x)g,(x), then we know the indefinite integral of g(x) f ,(x), and vice versa. To that end, we choose the first indefinite integral on the left in

Equation (5.7) and solve for it to generate the rule

r f (x)g,(x) dx = f (x)g(x) ? r g(x) f ,(x) dx. (5.8) Often we express Equation (5.8) in terms of the variables u and v, where u = f x and

v = g x . Note that in differential notation, du = f , x dx and dv = g, x dx, and thus we can state the rule for Integration by Parts in its most common form as follows.

To apply Integration by Parts, we look for a product of basic functions that we can identify as u and d . If we can antidifferentiate d to find , and evaluating du is not more difficult than evaluating u dv, then this substitution usually proves to be fruitful. To demonstrate, we consider the following example.

Example 5.3. Evaluate the indefinite integral

r x cos(x) dx

using Integration by Parts.

Solution. Whenever we are trying to integrate a product of basic functions through Integration by Parts, we are presented with a choice for u and dv. In the current problem, we can either let u = x and dv = cos x dx, or let u = cos x and dv = x dx. While there is not a universal rule for how to choose u and dv, a good guideline is this: do so in a way that v du is at least as simple as the original problem u dv.

In this setting, this leads us to choose6 u = x and dv = cos x dx, from which it follows that du = 1 dx and v = sin x . With this substitution, the rule for Integration by Parts tells us that

r x cos(x) dx = x sin(x) ? r sin(x) · 1 dx.

6Observe that if we considered the alternate choice, and let u = cos(x) and dv = x dx, then du =

? sin(x) dx and v = 1 x2, from which we would write

r x cos(x) dx = 1 x2 cos(x) ? r 1 x2(? sin(x)) dx.

Thus we have replaced the problem of integrating x cos x with that of integrating 1 x2 sin x ; the latter is clearly more complicated, which shows that this alternate choice is not as helpful as the first choice.

At this point, all that remains to do is evaluate the (simpler) integral sin x 1 dx. Doing so, we find

r x cos(x) dx = x sin(x) ? (? cos(x)) + C = x sin(x) + cos(x) + C.

There are at least two additional important observations to make from Example 5.3. First, the general technique of Integration by Parts involves trading the problem of integrating the product of two functions for the problem of integrating the product of two related functions. In particular, we convert the problem of evaluating u d for that of evaluating du. This perspective clearly shapes our choice of u and . In Example 5.3, the original integral to evaluate was x x dx, and through the substitution provided by Integration by Parts, we were instead able to evaluate sin x 1 dx. Note that the original function x was replaced by its derivative, while cos x was replaced by its antiderivative. Second, observe that when we get to the final stage of evaluating the last remaining antiderivative,

it is at this step that we include the integration constant, +C.

Activity 5.10.

Evaluate each of the following indefinite integrals. Check each antiderivative that you find by differentiating.

(a) (te?t dt

(b) (4x sin(3x) dx

(c) (z sec2(z) dz

(d) (x ln(x) dx

<1

Some Subtleties with Integration by Parts

There are situations where Integration by Parts is not an obvious choice, but the technique is appropriate nonetheless. One guide to understanding why is the observation that integration by parts allows us to replace one function in a product with its derivative while replacing the other with its antiderivative. For instance, consider the problem of evaluating r arctan(x) dx.

Initially, this problem seems ill-suited to Integration by Parts, since there does not appear to be a product of functions present. But if we note that arctan(x) = arctan(x) · 1, and realize that we know the derivative of arctan(x) as well as the antiderivative of 1, we

see the possibility for the substitution u = arctan x and dv = 1 dx. We explore this substitution further in Activity 5.11.

In a related problem, if we consider t3 sin t2 dt, two key observations can be made about the algebraic structure of the integrand: there is a composite function present in sin t2 , and there is not an obvious function-derivative pair, as we have t3 present (rather than simply t) multiplying sin t2 . This problem exemplifies the situation where we sometimes use both u-substitution and Integration by Parts in a single problem. If we write t3 = t · t2 and consider the indefinite integral

r t · t2 · sin(t2) dt,

we can use a mix of the two techniques we have recently learned. First, let z = t2 so that dz = 2t dt, and thus t dt = 1 dz. (We are using the variable z to perform a "z-substitution" since u will be used subsequently in executing Integration by Parts.) Under this z-substitution, we now have

r t · t2 · sin(t2) dt = r z · sin(z) · 1 dz.

The remaining integral is a standard one that can be evaluated by parts. This, too, is explored further in Activity 5.11.

The problems briefly introduced here exemplify that we sometimes must think creatively in choosing the variables for substitution in Integration by Parts, as well as that it is entirely possible that we will need to use the technique of substitution for an additional change of variables within the process of integrating by parts.

Activity 5.11.

Evaluate each of the following indefinite integrals, using the provided hints.

(a) Evaluate arctan(x) dx by using Integration by Parts with the substitution

u = arctan(x) and dv = 1 dx.

(b) Evaluate ln(z) dz. Consider a similar substitution to the one in (a).

(c) Use the substitution z = t2 to transform the integral t3 sin t2 dt to a new integral in the variable z, and evaluate that new integral by parts.

(d) Evaluate (s5es3 ds using an approach similar to that described in (c).

(e) Evaluate (e2t cos(et) dt. You will find it helpful to note that e2t = et · et .

Using Integration by Parts Multiple Times

We have seen that the technique of Integration by Parts is well suited to integrating the product of basic functions, and that it allows us to essentially trade a given integrand for a new one where one function in the product is replaced by its derivative, while the other is replaced by its antiderivative. The main goal in this trade of u dv for v du is to have the new integral not be more challenging to evaluate than the original one. At times, it turns out that it can be necessary to apply Integration by Parts more than once in order to ultimately evaluate a given indefinite integral.

For example, if we consider t2et dt and let u = t2 and dv = et dt, then it follows that

du = 2t dt and v = et , thus

r t2et dt = t2et ? r 2tet dt.

The integral on the righthand side is simpler to evaluate than the one on the left, but it still requires Integration by Parts. Now letting u = 2t and dv = et dt, we have du = 2 dt and v = et , so that r t2et dt = t2et ? i2tet ? r 2et dt\ .

Note the key role of the parentheses, as it is essential to distribute the minus sign to the entire value of the integral   2tet dt. The final integral on the right in the most recent equation is a basic one; evaluating that integral and distributing the minus sign, we find r t2et dt = t2et ? 2tet + 2et + C.

Of course, situations are possible where even more than two applications of Integration by Parts may be necessary. For instance, in the preceding example, it is apparent that if the integrand was t3et instead, we would have to use Integration by Parts three times.

Next, we consider the slightly different scenario presented by the definite integral

et cos(t) dt. Here, we can choose to let u be either et or cos(t); we pick u = cos(t), and thus dv = et dt. With du = ? sin(t) dt and v = et , Integration by Parts tells us that

r et cos(t) dt = et cos(t) ? r et (? sin(t)) dt,

or equivalently that

r et cos(t) dt = et cos(t) + r et sin(t) dt   (5.9)

Observe that the integral on the right in Equation (5.9),   et sin t dt, while not being more complicated than the original integral we want to evaluate, it is essentially identical

to   et cos t dt. While the overall situation isn't necessarily better than what we started with, the problem hasn't gotten worse. Thus, we proceed by integrating by parts again. This time we let u = sin(t) and dv = et dt, so that du = cos(t) dt and v = et , which implies r et cos(t) dt = et cos(t) + iet sin(t) ? r et cos(t) dt\   (5.10)

We seem to be back where we started, as two applications of Integration by Parts has led us back to the original problem, et cos t dt. But if we look closely at Equation (5.10), we see that we can use algebra to solve for the value of the desired integral. In particular, adding et cos(t) dt to both sides of the equation, we have

2 r et cos(t) dt = et cos(t) + et sin(t),

and therefore

r et cos(t) dt = 1 (et cos(t) + et sin(t)) + C.

Note that since we never actually encountered an integral we could evaluate directly, we didn't have the opportunity to add the integration constant C until the final step, at which point we include it as part of the most general antiderivative that we sought from the outset in evaluating an indefinite integral.

Activity 5.12.

Evaluate each of the following indefinite integrals.

(a) r x2 sin(x) dx

(b) r t3 ln(t) dt

(c) r ez sin(z) dz

(d) r s2e3s ds

(e) t arctan(t) dt

(Hint: At a certain point in this problem, it is very helpful to note that

2

1+t2

Evaluating Definite Integrals Using Integration by Parts

Just as we saw with u-substitution in Section 5.3, we can use the technique of Integration by Parts to evaluate a definite integral. Say, for example, we wish to find the exact value of

r ?/2

One option is to evaluate the related indefinite integral to find that t sin t dt = t cos t + sin t + C, and then use the resulting antiderivative along with the Fundamental Theorem of Calculus to find that

r ?/2

11?/2

t sin(t) dt = (?t cos(t) + sin(t))

= {? ? cos(?) + sin(?)(? (?0 cos(0) + sin(0))

2 2 2

= 1.

Alternatively, we can apply Integration by Parts and work with definite integrals throughout. In this perspective, it is essential to remember to evaluate the product uv over the given limits of integration. To that end, using the substitution u = t and dv = sin(t) dt, so that du = dt and v = ? cos(t), we write

?/2

0

t (t) dt ?t (t)111?/2 ?

?/2

0

(? cos(t)) dt

11?/2 11?/2

= {? ? cos(?) + sin(?)(? (?0 cos(0) + sin(0))

2 2 2

= 1.

As with any substitution technique, it is important to remember the overall goal of the problem, to use notation carefully and completely, and to think about our end result to ensure that it makes sense in the context of the question being answered.

When u-substitution and Integration by Parts Fail to Help

As we close this section, it is important to note that both integration techniques we have discussed apply in relatively limited circumstances. In particular, it is not hard to find examples of functions for which neither technique produces an antiderivative; indeed, there are many, many functions that appear elementary but that do not have an elementary

algebraic antiderivative. For instance, if we consider the indefinite integrals

r ex2 dx and r x tan(x) dx,

neither u-substitution nor Integration by Parts proves fruitful. While there are other integration techniques, some of which we will consider briefly, none of them enables us to find an algebraic antiderivative for ex2 or x tan x . There are at least two key observations to make: one, we do know from the Fundamental Theorem Part 2 of Calculus that we can construct an integral antiderivative for each function; and two, antidifferentiation is much, much harder in general than differentiation. In particular, we observe that F(x) = x et2 dt is an antiderivative of f (x) = ex2 , and G(x) = x t tan(t) dt

is an antiderivative of g x = x tan x . But finding an elementary algebraic formula that doesn’t involve integrals for either F or G turns out not only to be impossible through u-substitution or Integration by Parts, but indeed impossible altogether.

Summary

In this section, we encountered the following important ideas:

• Through the method of Integration by Parts, we can evaluate indefinite integrals that involve products of basic functions such as x sin x dx and x ln x dx through a substitution that enables us to effectively trade one of the functions in the product for its derivative, and the other for its antiderivative, in an effort to find a different product of functions that is easier to integrate.

• If we are given an integral whose algebraic structure we can identify as a product of basic functions in the form f (x)g,(x) dx, we can use the substitution u = f (x) and dv = g,(x) dx and apply the rule

r u dv = uv ? r v du

in an effort to evaluate the original integral (f (x)g,(x) dx by instead evaluating

• When deciding to integrate by parts, we normally have a product of functions present in the integrand and we have to select both u and dv. That selection is guided by the overall principal that we desire the new integral du to not be any more difficult or complicated than the original integral u dv. In addition, it is often helpful to recognize if one of the functions present is much easier to differentiate than antidifferentiate (such as ln x), in which case that function often is best assigned the variable u. For sure, when choosing dv, the corresponding function must be one that we can antidifferentiate.

Exercises

1. Let f (t) = te?2t and F(x) = (x f (t) dt.

(a) Determine F ,(x).

(b) Use the FTC 1 to find a formula for F that does not involve an integral.

(c) Is F an increasing or decreasing function for x > 0? Why?

2. Consider the indefinite integral given by   e2x cos(ex) dx.

(a) Noting that e2x = ex   ex , use the substitution z = ex to determine a new, equivalent integral in the variable z.

(b) Evaluate the integral you found in (a) using an appropriate technique.

(c) How is the problem of evaluating e2x cos e2x dx different from evaluating the integral in (a)? Do so.

(d) Evaluate each of the following integrals as well, keeping in mind the approach(es) used earlier in this problem:

- (e2x sin(ex) dx
- xex2 cos(ex2) sin(ex2) dx

3. For each of the following indefinite integrals, determine whether you would use u-substitution, integration by parts, neither*, or both to evaluate the integral. In each case, write one sentence to explain your reasoning, and include a statement of any substitutions used. (That is, if you decide in a problem to let u = e3x , you should state that, as well as that du = 3e3x dx.) Finally, use your chosen approach to evaluate each integral. (* one of the following problems does not have an elementary antiderivative and you are not expected to actually evaluate this integral; this will correspond with a choice of "neither" among those given.)

(a) (x2 cos(x3) dx

(b) (x5 cos(x3) dx (Hint: x5 = x2 · x3)

(c) (x ln(x2) dx

(d) (sin(x4) dx

(e) (x3 sin(x4) dx

(f) (x7 sin(x4) dx

5.5 Other Options for Finding Algebraic Antiderivatives

Introduction

In the preceding sections, we have learned two very specific antidifferentiation techniques: u-substitution and integration by parts. The former is used to reverse the chain rule, while the latter to reverse the product rule. But we have seen that each only works in very specialized circumstances. For example, while xex2 dx may be evaluated by

u-s ubstitution and xex dx by integration by parts, neither method provides a route to evaluate ex2 dx. That fact is not a particular shortcoming of these two antidifferentiation techniques, as it turns out there does not exist an elementary algebraic antiderivative for

ex2 . Said differently, no matter what antidifferentiation methods we could develop and

learn to execute, none of them will be able to provide us with a simple formula that does not involve integrals for a function F(x) that satisfies F ,(x) = ex .

In this section of the text, our main goals are to better understand some classes of functions that can always be antidifferentiated, as well as to learn some options for so doing. At the same time, we want to recognize that there are many functions for which an algebraic formula for an antiderivative does not exist, and also appreciate the role that computing technology can play in helping us find antiderivatives of other complicated functions. Throughout, it is helpful to remember what we have learned so far: how to reverse the chain rule through u-substitution, how to reverse the product rule through integration by parts, and that overall, there are subtle and challenging issues to address when trying to find antiderivatives.

Preview Activity 5.5. For each of the indefinite integrals below, the main question is to decide whether the integral can be evaluated using u-substitution, integration by parts, a combination of the two, or neither. For integrals for which your answer is affirmative, state the substitution(s) you would use. It is not necessary to actually evaluate any of the integrals completely, unless the integral can be evaluated immediately using a familiar

basic antiderivative.

(a) r x2 sin(x3) dx, r x2 sin(x) dx, r sin(x3) dx, r x5 sin(x3) dx

1

(b) 1 + x2 dx,

x

1 + x2 dx,

2x + 3

1 + x2 dx,

ex

,

1 + (ex)2

(c) r x ln(x) dx, r ln(x) dx, r ln(1 + x2) dx, r x ln(1 + x2) dx,

(d) r x?1 ? x2 dx, r ? 1 dx, r ? x dx, r ? 1 dx,

1 ? x2

1 ? x2

x 1 ? x2

t><I

The Method of Partial Fractions

The method of partial fractions is used to integrate rational functions, and essentially involves reversing the process of finding a common denominator. For example, suppoes we have the function R x = 5x and want to evaluate

x2?x?2

5x dx. x2 ? x ? 2

Thinking algebraically, if we factor the denominator, we can see how R might come from the sum of two fractions of the form A + B . In particular, suppose that

x?2

5x

x+1

A B

= + .

(x ? 2)(x + 1)

x ? 2 x + 1

Multiplying both sides of this last equation by (x ? 2)(x + 1), we find that

5x = A(x + 1) + B(x ? 2).

Since we want this equation to hold for every value of x, we can use insightful choices of specific x-values to help us find A and B. Taking x = ?1, we have

5(?1) = A(0) + B(?3),

and thus B = 5 . Choosing x = 2, it follows

5(2) = A(3) + B(0),

so A = 10 . Therefore, we now know that

r 5x dx = r 10/3 + 5/3 dx.

x2 ? x ? 2

x ? 2

x + 1

This equivalent integral expression is straightforward to evaluate, and hence we find that

r 5x dx = 10 ln |x ? 2| + 5 ln |x + 1| + C.

x2 ? x ? 2 3 3

It turns out that for any rational function R(x) = where the degree of the polynomial

P is less than7

Q(x)

the degree of the polynomial Q, the method of partial fractions can be

used to rewrite the rational function as a sum of simpler rational functions of one of the following forms:

A

,

x ? c

A

, or

(x ? c)n

Ax + B

x2 + k

where A, B, and c are real numbers, and k is a positive real number. Because each of these basic forms is one we can antidifferentiate, partial fractions enables us to antidifferentiate any rational function.

A computer algebra system such as Maple, Mathematica, or WolframAlpha can be used to find the partial fraction decomposition of any rational function. In WolframAlpha, entering

partial fraction 5x/(xˆ2-x-2)

results in the output

5x 10 5

= + .

x2 ? x ? 2 3(x ? 2) 3(x + 1)

We will primarily use technology to generate partial fraction decompositions of rational functions, and then work from there to evaluate the integrals of interest using established methods.

Activity 5.13.

For each of the following problems, evaluate the integral by using the partial fraction decomposition provided.

(a) r 1 dx, given that 1 = 1/4 ? 1/4

x2 ? 2x ? 3

x2?2x?3

x?3

x+1

(b) r x2 + 1

dx, given that x2+1

? 1 ? 1 2

x3 ? x2

x3?x2 = x

x2 + x?1

(c) r x ? 2 dx, given that x?2 = 1 ? 2 + ?x+2

<1

[7] If the degree of P is greater than or equal to the degree of Q, long division may be used to write R as the sum of a polynomial plus a rational function where the numerator's degree is less than the denominator's.

Using an Integral Table

Calculus has a long history, with key ideas going back as far as Greek mathematicians in 400-300 BC. Its main foundations were first investigated and understood independently by Isaac Newton and Gottfried Wilhelm Leibniz in the late 1600s, making the modern ideas of calculus well over 300 years old. It is instructive to realize that until the late 1980s, the personal computer essentially did not exist, so calculus (and other mathematics) had to be done by hand for roughly 300 years. During the last 30 years, however, computers have revolutionized many aspects of the world we live in, including mathematics. In this section we take a short historical tour to precede the following discussion of the role computer algebra systems can play in evaluating indefinite integrals. In particular, we consider a class of integrals involving certain radical expressions that, until the advent of computer algebra systems, were often evaluated using an integral table.

As seen in the short t?able of integral?s found in Appendix A, there are also many forms

of integrals that involve a2 ± w2 and w2 ? a2. These integral rules can be developed

using a technique known as trigonometric substitution that we choose to omit; instead, we

will simply accept the results presented in the table. To see how these rules are needed and used, consider the differences among

r ? 1 dx, r ? x

dx, and r ?1 ? x2 dx.

1 ? x2 1 ? x2

The first integral is a familiar basic one, and results in arcsin x + C. The second integral can be evaluated using a standard u-substitution with u = 1 x2. The third, however, is not familiar and does not lend itself to u-substitution.

In Appendix A, we find the rule

(8) a2 ? u2 du =

u ?

2

a2 u

arcsin

2 a

+ C.

Using the substitutions a = 1 and u = x (so that du = dx), it follows that

r ?1 ? x2 dx = x ?1 ? x2 ? 1 arcsin x + C.

One important point to note is that whenever we are applying a rule in the table, we are doing a u-substitution. This is especially key when the situation is more complicated than allowing u = x as in the last example. For instance, say we wish to evaluate the integral

r ?9 + 64x2 dx.

Once again, we want to use Rule (3) from the table, but now do so with a = 3 and u = 8x; we also choose the "+" option in the rule. With this substitution, it follows that du = 8dx,

so dx = 1 . Applying this substitution,

r ?9 + 64x2 dx = r ?9 + u2 · 1 du = 1 r ?9 + u2 du.

8 8

By Rule (3), we now find that

r ?9 + 64x2 dx = 1 { u ?u2 + 9 + 9 ln |u + ?u2 + 9| + C(

= 64x2 + 9 + ln |8x + ?64x2 + 9| + C(.

In problems such as this one, it is essential that we not forget to account for the factor of

1 that must be present in the evaluation.

Activity 5.14.

For each of the following integrals, evaluate the integral using u-substitution and/or an entry from the table found in Appendix A.

(a) r ?x2 + 4 dx

(b) r

(c) r

x

?x2 + 4 dx

2

? dx

(d) r ? 1 dx

x2 49 ? 36x2

<1

Using Computer Algebra Systems

A computer algebra system (CAS) is a computer program that is capable of executing symbolic mathematics. For a simple example, if we ask a CAS to solve the equation ax2 + bx + c = 0 for the variable x, where a, b, and c are arbitrary constants, the program

will return x = ?b±?b2?4ac . While research to develop the first CAS dates to the 1960s,

these programs became more common and publicly available in the early 1990s. Two prominent early examples are the programs Maple and Mathematica, which were among the first computer algebra systems to offer a graphical user interface. Today, Maple and Mathematica are exceptionally powerful professional software packages that are capable of executing an amazing array of sophisticated mathematical computations. They are also very expensive, as each is a proprietary program. The CAS SAGE is an open-source, free alternative to Maple and Mathematica.

For the purposes of this text, when we need to use a CAS, we are going to turn instead to a similar, but somewhat different computational tool, the web-based “computational knowledge engine” called WolframAlpha. There are two features of WolframAlpha that make it stand out from the CAS options mentioned above: (1) unlike Maple and Math- ematica, WolframAlpha is free (provided we are willing to suffer through some pop-up advertising); and (2) unlike any of the three, the syntax in WolframAlpha is flexible. Think of WolframAlpha as being a little bit like doing a Google search: the program will interpret what is input, and then provide a summary of options.

If we want to have WolframAlpha evaluate an integral for us, we can provide it syntax such as integrate   xˆ2 dx

to which the program responds with

r x2 dx =

x3   constant

3

While there is much to be enthusiastic about regarding CAS programs such as Wolfra- mAlpha, there are several things we should be cautious about: (1) a CAS only responds to exactly what is input; (2) a CAS can answer using powerful functions from highly advanced mathematics; and (3) there are problems that even a CAS cannot do without additional human insight.

Although (1) likely goes without saying, we have to be careful with our input: if we enter syntax that defines a function other than the problem of interest, the CAS will work with precisely the function we define. For example, if we are interested in evaluating the integral

and we mistakenly enter

integrate 1/16 - 5xˆ2 dx

a CAS will (correctly) reply with

1 dx,

16 ? 5x2

1 x ? 5 x3.

It is essential that we are sufficiently well-versed in antidifferentiation to recognize that this

function cannot be the one that we seek: integrating a rational function such as
1 2 ,

we expect the logarithm function to be present in the result.

Regarding (2), even for a relatively simple integral such as (1

16?5x

2 dx, some CASs

will invoke advanced functions rather than simple ones. For execute the command

16?5x we use Maple to

int(1/(16-5*xˆ2), x);

the program responds with

r 1 dx = ?5 arctanh(?5 x).

16 ? 5x2 20 4

While this is correct (save for the missing arbitrary constant, which Maple never reports), the inverse hyperbolic tangent function is not a common nor familiar one; a simpler way to express this function can be found by using the partial fractions method, and happens to be the result reported by WolframAlpha:

r 1 dx = 1 (log(4?5 + 5?x) ? log(4?5 ? 5?x)1 + constant.

Using sophisticated functions from more advanced mathematics is sometimes the way a CAS says to the user "I don't know how to do this problem." For example, if we want to evaluate

r e?x2 dx,

and we ask WolframAlpha to do so, the input

integrate exp(-xˆ2) dx

results in the output

r e?x2 dx =

?

erf x + constant.

2

The function “erf(x)” is the error function, which is actually defined by an integral:

2 x

erf(x) = ??

e?t2

dt.

So, in producing output involving an integral, the CAS has basically reported back to us the very question we asked.

Finally, as remarked at (3) above, there are times that a CAS will actually fail without some additional human insight. If we consider the integral

r (1 + x)ex ?1 + x2e2x dx

and ask WolframAlpha to evaluate

int (1+x) * exp(x) * sqrt(1+x^2 * exp(2x)) dx, the program thinks for a moment and then reports

(no result found in terms of standard mathematical functions)

But in fact this integral is not that difficult to evaluate. If we let u = xex , then du =

x

(1 + x)e

dx, which means that the preceding integral has form

r (1 + x)ex ?1 + x2e2x dx = r ?1 + u2 du,

which is a straightforward one for any CAS to evaluate.

So, the above observations regarding computer algebra systems lead us to proceed with some caution: while any CAS is capable of evaluating a wide range of integrals (both definite and indefinite), there are times when the result can mislead us. We must think carefully about the meaning of the output, whether it is consistent with what we expect, and whether or not it makes sense to proceed.

Summary

In this section, we encountered the following important ideas:

• The method of partial fractions enables any rational function to be antidifferentiated, because any polynomial function can be factored into a product of linear and irreducible quadratic terms. This allows any rational function to be written as the sum of a polynomial plus rational terms of the form A n (where n is a natural number) and

(x?c)

Bx+C (where k is a positive real number).

• Until the development of computing algebra systems, integral tables enabled students of calculus to more easily evaluate integrals such as a2 + u2 du, where a is a positive real number. A short table of integrals may be found in Appendix A.

• Computer algebra systems can play an important role in finding antiderivatives, though we must be cautious to use correct input, to watch for unusual or unfamiliar advanced functions that the CAS may cite in its result, and to consider the possibility that a CAS may need further assistance or insight from us in order to answer a particular question.

Exercises

1. For each of the following integrals involving rational functions, (1) use a CAS to find the partial fraction decomposition of the integrand; (2) evaluate the integral of the resulting function without the assistance of technology; (3) use a CAS to evaluate the original integral to test and compare your result in (2).

(a) x3 + x + 1 dx x4 ? 1

(b) x5 + x2 + 3 dx x3 ? 6x2 + 11x ? 6

(c) x2 ? x ? 1 dx

(x ? 3)3

2. For each of the following integrals involving radical functions, (1) use an appropriate u-substitution along with Appendix A to evaluate the integral without the assistance of technology, and (2) use a CAS to evaluate the original integral to test and compare your result in (1).

(a) r ? 1 dx

(b) r x?1 + x4 dx

(c) r ex ?4 + e2x dx

(d) r

tan(x) dx

9 ? cos2(x)

3. Consider the indefinite integral given by

x + ?1 + x2

x dx.

(a) Explain why u-substitution does not offer a way to simplify this integral by discussing at least two different options you might try for u.

(b) Explain why integration by parts does not seem to be a reasonable way to proceed, either, by considering one option for u and dv.

(c) Is there any line in the integral table in Appendix A that is helpful for this integral?

(d) Evaluate the given integral using WolframAlpha. What do you observe?

I. TI – Limits Basics – at end (opt.)

II. GeoGebra apps

1. https://ggbm.at/K8MnDYh4

IV. Practice – Khan Academy

1. Complete the six online practice exercises in the first unit (Continuity) of Khan Academy's AP Calculus AB course: https://www.khanacademy.org/math/ap-calculus-ab/continuity-ab?t=practice