3.3B A Deeper Dive: FTC Part 2

Finding Antiderivatives and Evaluating Integrals

In Lesson 3.3A, you learned the Fundamental Theorem of Calculus (FTC), which from here forward will be referred to as the Fundamental Theorem of Calculus Part 1[1], as in this section we develop a corresponding result. In particular, recall that the FTC 1 tells us that if f is a continuous function on a, b and F is any antiderivative of f (that is, $F' = f$), then

$$\int_a^b f(x)dx = F(b) - F(a)$$

You have used this result in two settings: (1) where f is a function whose graph we know and for which we can compute the exact area bounded by f on a certain interval $[a, b]$, we can compute the change in an antiderivative F over the interval; and (2) where f is a function for which it is easy to determine an algebraic formula for an antiderivative, we may evaluate the integral exactly and hence determine the net-signed area bounded by the function on the interval.

Investigation 1: First FTC Redux

Consider the function A defined by the rule

$$A(x) = \int_1^x f(t)dt,$$

where $f(t) = 4 - 2t$.

1a) Compute $A(1)$ and $A(2)$ exactly.

[1] On the AP exam, there is no need to distinguish between FTC, Part 1 and FTC Part 2. There is only one Theorem, with different applications.

b) Use the Fundamental Theorem Part 1 of Calculus to find an equivalent formula for $A(x)$ that does not involve integrals. That is, use FTC 1 to evaluate $\int_1^x (4-2t)dt$.

c) Observe that f is a linear function; what kind of function is A?

d) Using the formula you found in (b) that does not involve integrals, compute $A'(x)$.

e) While you have defined f by the rule $f(t) = 4 - 2t$, it is equivalent to say that f is given by the rule $f(x) = 4 - 2x$. What do you observe about the relationship between A and f?

II. The Fundamental Theorem Part 2 of Calculus

The result of Investigation 1 is not particular to the function $f(t) = 4 - 2t$, nor to the choice of "1" as the lower bound in the integral that defines the function A. For instance, if we let $f(t) = cost(t) - t$ and set $A(x) = \int_2^x f(t)dt$, then we can determine a formula for A without integrals by the FTC 1. Specifically,

$$A(x) = \int_2^x (\cos(t) - t)\, dt$$

$$= \sin(t) - \frac{1}{2}t^2 |_2^x$$

$$= \sin(x) - \frac{1}{2}x^2 - (\sin(2) - 2).$$

Differentiating $A(x)$, since (sin (2) - 2) is constant, it follows that

$$A'(x) = \cos(x) - x,$$

and thus we see that $A'(x) = f(x)$. This tells us that for this particular choice of f, A is an antiderivative of f. More specifically, since $A(2) = \int_2^2 f(t)dt = 0$, A is the only antiderivative of f for which $A(2) = 0$.

In general, if f is any continuous function, and we define the function A by the rule

$$A(x) = \int_c^x f(t)dt$$

where c is an arbitrary constant, then we can show that A is an antiderivative of f. To see why, let's demonstrate that $A'\ x = f\ (x)$ by using the limit definition of the derivative. Doing so, we observe that

$$A'(x) = \lim_{h\to\infty} \frac{A(x+h) - A(x)}{h}$$

$$= \lim_{x\to\infty} \frac{\int_c^{x+h} f(t)dt - \int_c^x f(t)dt}{h}$$

$$= \lim_{x\to\infty} \frac{\int_c^{x+h} f(t)dt}{h}$$

where the last equation in the preceding chain follows from the fact $\int_c^x f(t)dt + \int_x^{x+h} f(t)dt = \int_c^{x+h} f(t)dt$. Now, observe that for small values of h,

$$\int_x^{x+h} f(t)dt \approx f(x)\cdot h$$

by a simple left-hand approximation of the integral. Thus, as we take the limit, it follows that

$$A"(x) = \lim_{h\to 0} \frac{\int_c^{x+h} f(t)dt}{h} = \lim_{h\to 0} \frac{f(x)\cdot h}{h} = f(x)$$

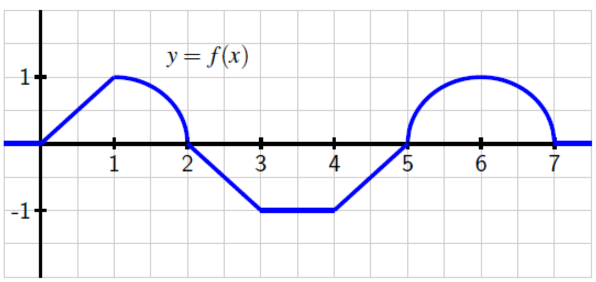
Hence, A is indeed an antiderivative of f. In addition, $A(c) = \int_c^c f(t)dt = 0$. The preceding argument demonstrates the truth of the Fundamental Theorem of Calculus Part 2, which we state as follows.

The Fundamental Theorem of Calculus, Part 2: If f is a continuous function, and c is any constant , then f has a unique antiderivative that satisfies $A(c) = 0$, and that antiderivative is given by the rule

$$A(x) = \int_c^x f(t)dt$$

Investigation 2: FTC 2

2.Suppose that f is the function graphed below and that f is a piecewise function whose parts are either portions of lines or portions of circles, as pictured. In addition,



a) What does the FTC 2 tell us about the relationship between A and f ?

b) Compute $A(1)$ and $A(3)$ exactly.

c) Sketch a precise graph of $y = A\ (x)$ on the axes at right that accurately reflects where A is increasing and decreasing, where A is concave up and concave down, and the exact values of A at $x = 0, 1, \ldots, 7$.

d) How is A similar to, but different from, the function F that you found in Investigation 1?

e) With as little additional work as possible, sketch precise graphs of the functions

$B(x) = \int_3^x f(t)dt$ and $B(x) = \int_1^x f(t)dt$. Justify *your* results with at least one sentence of explanation.

III. Understanding (and Describing) Integral Functions

The FTC 2 provides a means to construct an antiderivative of any continuous function. If given a continuous function g and wish to find an antiderivative of G, we can now say that

$$G(x) = \int_c^x g(t)\, dt$$

provides the rule for such an antiderivative, and moreover that $G(c) = 0$. Note especially that we know that $G'(x) = g(x)$. We sometimes want to write this relationship between G and g from a different notational perspective. In particular, observe that

$$\frac{d}{dx}\left[\int_c^x g(t)dt\right] = g(x) \qquad \textbf{Equation 1}$$

This result can be particularly useful when we're given an integral function such as G and wish to understand properties of its graph by recognizing that $G'(x) = g(x)$, while not necessarily being able to exactly evaluate the definite integral $\int_c^x g(t)dt$.

Example 1. Investigate the behavior of the integral function

$$E(x) = \int_0^x e^{-t^2} dt\ .$$

E is closely related to the well-known error function[2], a function that is particularly important in probability and statistics.

---

[2] The error function is defined by the rule $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2} dt$ and has the key property that $0 \le \operatorname{erf}(x) < 1$for all $x > 0$ and moreover that $\lim_{x\to\infty} \operatorname{erf}(x) = 1$.

Solution : While you cannot evaluate E exactly for any value other than $x = 0$, you still can gain a tremendous amount of information about the function E. To begin, applying the rule in Equation 1 to E, it follows that

$$E'(x) = \frac{d}{dx}\left[\int_0^x e^{-t^2}dt\right] = e^{-x^2},$$

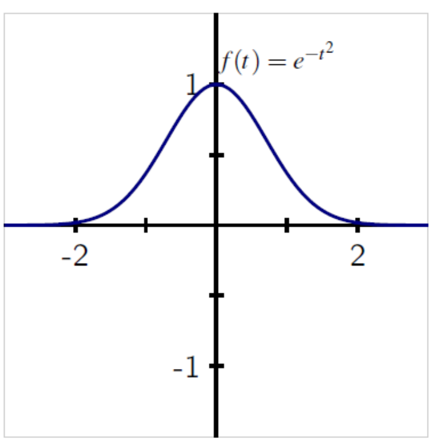
so you know a formula for the derivative of E. Moreover, you know that $E(0) = 0$.

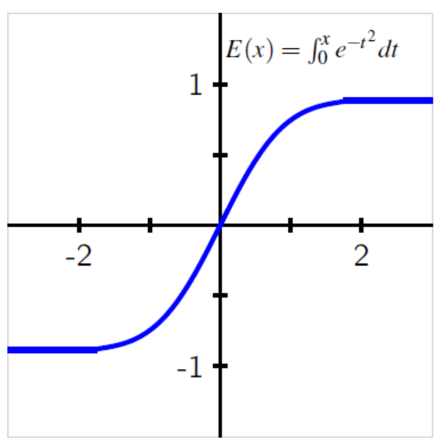
Using the first and second derivatives of E, along with the fact that $E(0) = 0$, you can determine more information about the behavior of E. First, with $E'(x) = e^{-x^2}$, note that for all real numbers x, $e^{-x^2} > 0$, and thus $E'(x) > 0$ for all x. Thus <u>E is an always increasing function</u>.

Further, note that as $x \to \infty$, $E'(x) = e^{-x^2} \to \infty$, hence the slope of the function E tends to zero as x (and similarly as $x \to -\infty$). Indeed, it turns out that <u>E has horizontal asymptotes as x increases or decreases without bound</u>.

In addition, observe that $E''(x) = -2xe^{-x^2}$, and that $E''(0) = 0$, while $E''(x) < 0$ for $x > 0$ and $E''(x) > 0$ for $x < 0$. This information tells us that <u>E is concave up for $x < 0$ and concave down for $x > 0$ with a point of inflection at $x = 0$</u>.

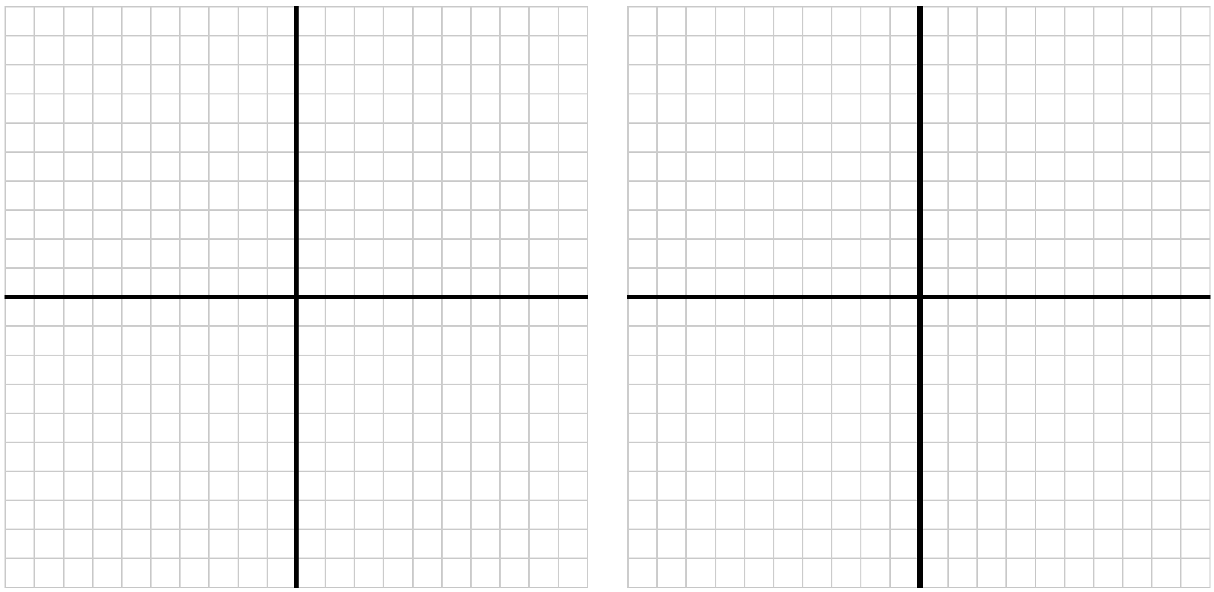
The only thing lacking is a sense of how big E can get as x increases. If you use a midpoint Riemann sum with 10 subintervals to estimate $E(2)$, we see that $E(2) \approx 0.8822$; a similar calculation to estimate $E(3)$ shows little change ($E(3) \approx 0.8862$), so it appears that as x increases without bound, E approaches a value just larger than 0.886, which aligns with the fact that E has horizontal asymptotes. Putting all of this information together (and using the symmetry of $f(t) = e^{-t^2}$), you can see the results shown below.





Investigation 3:

Suppose that $f(t) = \frac{t}{1+t^2}$ and $F(x) = \int_0^x f(t)\, dt$.



3a) On the axes at left in Figure 5.12, plot a graph of $f(t) = \frac{t}{1+t^2}$ on the interval $-10 \leq t \leq 10$. Clearly label the vertical axes with appropriate scale.

b) What is the key relationship between F and f, according to the FTC 2?

c) Use the first derivative test to determine the intervals on which F is increasing and decreasing.

d) Use the second derivative test to determine the intervals on which F is concave up and concave down. Note that $f'(t)$ can be simplified to be written in the form $f'(t) = \frac{1-t^2}{(1+t^2)^2}$.

e) Using technology appropriately, estimate the values of $F(5)$ and $F(10)$ through appropriate Riemann sums.

f) Sketch an accurate graph of $y = F(x)$ on the right-hand axes provided, and clearly label the vertical axes with appropriate scale.

IV. Differentiating an Integral Function

You have seen that the FTC 2 enables you to construct an antiderivative F of any continuous function f by defining F by the corresponding integral function $F(x) = \int_x^c f(t)dt$. This means that integral functions, which may appear complicated, are nonetheless particularly simple to differentiate. For instance, if

$$F(x) = \int_{\pi}^{x} \sin(t^2)\, dt$$

then by the FTC 2, you know that

$$F'(x) = \sin(x^2).$$

Stating this result more generally for an arbitrary function f, you know by the FTC 2 that

$$\frac{d}{dx}\left[\int_{x}^{a} f(t)dt\right] = f(x)$$

In words, the last equation essentially says that "the derivative of the integral function whose integrand is f' is f." In this sense, we see that if we first integrate the function f from $t = a$ to $t = x$, and then differentiate with respect to x, these two processes "undo" one another.

Taking a different approach, say we begin with a function $f(t)$ and differentiate with respect to t. What happens if we follow this by integrating the result from $t = a$ to $t = x$?

That is, what can we say about the quantity

$$\int_a^x \frac{dy}{dx}[f(t)]\,dt\ ?$$

Here, we use FTC 1 and note that $f(t)$ is an antiderivative of $\frac{d}{dt}[(f(t)]$. Applying this result and evaluating the antiderivative function, you see that

$$\int_a^x \frac{d}{dt}[f(t)dt] = f(t)\Big]_a^x$$

$$= \text{f(x)} - \text{f(a)}$$

Thus, we see that if we apply the processes of first differentiating f and then integrating the result from a to x, we return to the function f ' minus the constant value $f(a)$. So in this situation, the two processes almost undo one another, up to the constant $f(a)$.

This demonstrate that differentiating and integrating (where we integrate from a constant up to a variable) are almost inverse processes. In one sense, this should not be surprising: integrating involves antidifferentiating, which reverses the process of differentiating. On the other hand, we see that there is some subtlety involved, as integrating the derivative of a function does not quite produce the function itself. This is connected to a key fact we observed earlier: that any function has an entire family of antiderivatives, and any two of those antiderivatives differ only by a constant.

Investigation 4:

4. Evaluate each of the following derivatives and definite integrals. Clearly state whether you used FTC 1 or FTC 2 in so doing.

a) $\frac{d}{dx}\left[\int_4^x e^{t^2}\,dt\right]$

b) $\int_{-2}^x \frac{d}{dt}\left[\frac{t^4}{1+t^4}\right]dt$

c) $\dfrac{d}{dx}\left[\displaystyle\int_x^1 \cos(t^3)dt\right]$

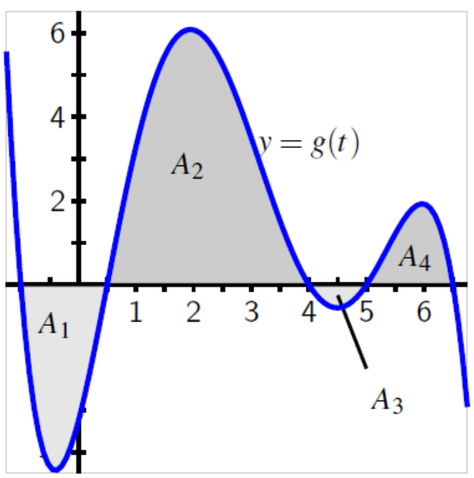
d) $\displaystyle\int_3^x \dfrac{d}{dt}[\ln(1+t^2)]\,dt$

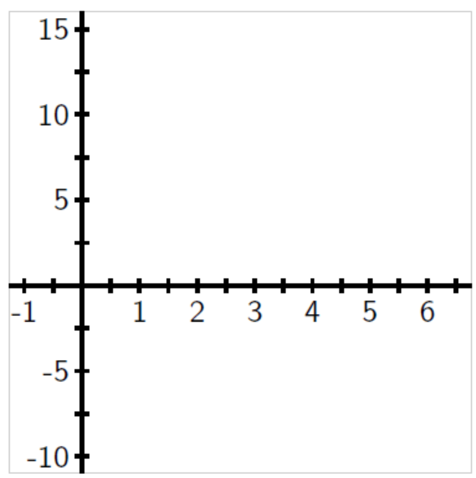
e) $\dfrac{d}{dx}\left[\displaystyle\int_4^{x^3} \sin(t^2)dt\right]$

V. Practice

1. Let g be the function pictured below left, and let F be defined by $F(x) = \int_2^x g(t)dt$. Assume that the shaded areas have values $A_1 = 4.29$, $A_2 = 12.75$, $A_3 = 0.36$ and $A_4 = 1.79$. Assume further that the portion of A_2 that lies between $x = 0.5$ and $x = 2$ is 6.06.

Sketch a carefully labeled graph of F on the axes provided, and include a written analysis of how you know where F is zero, increasing, decreasing, CCU, and CCD.





2. The tide removes sand from the beach at a small ocean park at a rate modeled by the function

$$R(t) = 2 + 5\sin(\frac{4\pi t}{25})$$

A pumping station adds sand to the beach at rate modeled by the function

$$S(t) = \frac{15t}{1 + 3t}$$

Both $R(t)$ and $S(t)$ are measured in cubic yards of sand per hour, t is measured in hours, and the valid times are $0 \le t \le 6$. At time $t = 0$, the beach holds 2500 cubic yards of sand.

a) What definite integral measures how much sand the tide will remove during the time period $0 \le t \le 6$? Why?

b) Write an expression for $Y(x)$, the total number of cubic yards of sand on the beach at time x. Carefully explain your thinking and reasoning.

c) At what instantaneous rate is the total number of cubic yards of sand on the beach at time $t = 4$ changing?

d) Over the time interval $0 \le t \le 6$, at what time t is the amount of sand on the beach least? What is this minimum value? Explain and justify your answers fully.

3. When an aircraft attempts to climb as rapidly as possible, its climb rate (in feet per minute) decreases as altitude increases, because the air is less dense at higher altitudes. Given below is a table showing performance data for a certain single engine aircraft, giving its climb rate at various altitudes, where $c(h)$ denotes the climb rate of the airplane at an altitude h.

h, feet	0	1000	2000	3000	4000	5000	6000	7000	8000	9000	10000
c (ft/min)	925	875	830	780	730	685	635	585	535	490	440

Let a new function m, that also depends on h, (say $y = m(h)$) measure the number of minutes required for a plane at altitude h to climb the next foot of altitude.

a. Determine a similar table of values for $m(h)$ and explain how it is related to the table above. Be sure to discuss the units on m.

b. Give a careful interpretation of a function whose derivative is $m(h)$. Describe what the input is and what the output is. Also, explain in plain English what the function tells us.

c. Determine a definite integral whose value tells us exactly the number of minutes required for the airplane to ascend to 10,000 feet of altitude. Clearly explain why the value of this integral has the required meaning.

d. Determine a formula for a function $M(h)$ whose value tells us the exact number of minutes required for the airplane to ascend to h feet of altitude.

e. Estimate the values of $M(6000)$ and $M(10{,}000)$ as accurately as you can. Include units on your results.

VI. Practice – Khan Academy

Complete the following online practice exercise in the FTC unit of Khan Academy's AP Calculus AB course:

1. https://www.khanacademy.org/math/ap-calculus-ab/fundamental-theorem-of-calculus-ab?t=practice
2. https://www.khanacademy.org/math/ap-calculus-ab/fundamental-theorem-of-calculus-ab/fundamental-theorem-of-calculus-chain-rule-ab/e/second-fundamental-theorem-of-calculus