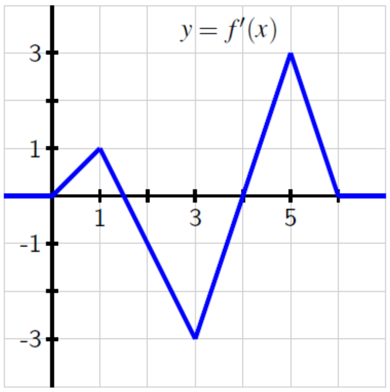
3.3B A Deeper Dive

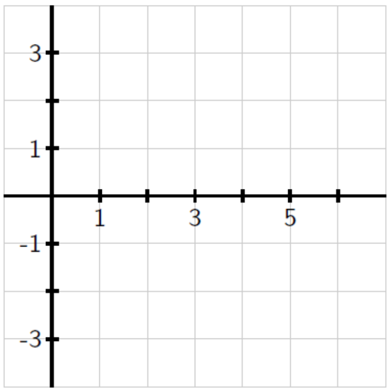
Finding Antiderivatives and Evaluating Functions Defined by Integrals

Over the past few lessons, you discovered key connections between the net-signed area under the velocity function and the corresponding change in position of the function. In Lesson 3.3A, the Total Change Theorem further illuminated these connections between f' and f in a more general setting, such as the one found in Figure 4.34, showing that the total change in the value of f over an interval a, b is determined by the exact net-signed area bounded by f' and the x-axis on the same interval.

In this lesson, you explore these issues still further, with a particular emphasis on situations where an accurate graph of the derivative function is available, along with a single value of the function f. From that information, we desire to completely determine an accurate graph of f that not only represents correctly where f is increasing, decreasing, concave up, and concave down, but also allows us to find an accurate function value at any point of interest to us.

Investigation 1: Suppose that the following information is known about a function f: the graph of its derivative, $y = f'(x)$, is given below. Further, assume that f' is piecewise linear (as pictured) and that for $x \leq 0$ and $x \geq 6$, $f'(x) = 0$. Finally, it is given that $f(0) = 1$.





1a) On what interval(s) is f an increasing function? On what intervals is f decreasing?

b) On what interval(s) is f concave up? concave down?

c) At what point(s) does f have a relative minimum? a relative maximum?

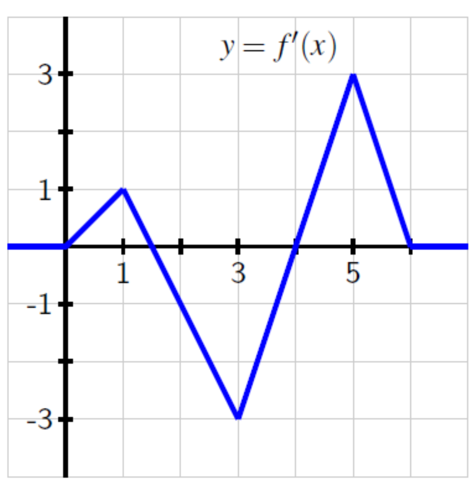
d) Recall that the Total Change Theorem tells us that

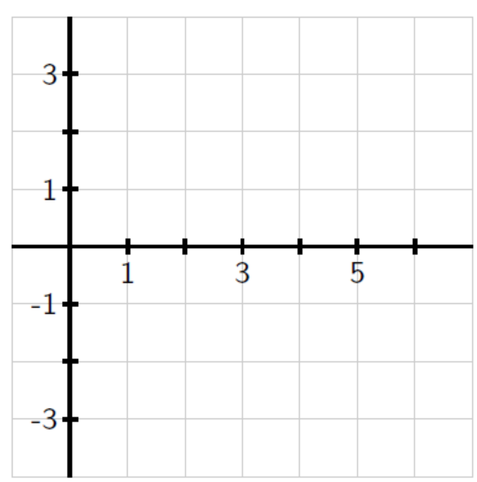
$$f(1) - f(0) = \int_0^1 f'(x)\, dx$$

What is the exact value of $f(1)$?

e) Use the given information and similar reasoning to that in (d) to determine the exact value of $f(2)$, $f(3)$, $f(4)$, $f(5)$, and $f(6)$.

f) Based on your responses to all of the preceding questions, sketch a complete and accurate graph of $y = f(x)$ on the axes provided, being sure to indicate the behavior of f for $x < 0$ and $x > 6$.





II. Constructing the graph of an antiderivative

Investigation 1 demonstrates one consequence of the Fundamental Theorem of Calculus: if you know a function f and wish to know information about its antiderivative, F, provided that we have some starting point a for which we know the value of $F(a)$, we can determine the value of $F(b)$ via the definite integral. In particular, since $F(b) - F(a) = \int_a^b f(x)dx$, it follows that

$$F(b) = F(a) + \int_a^b f(x)\, dx \qquad \textbf{Equation 1}$$

Rephrasing this in terms of a given function f and its antiderivative F, we observe that on an interval $[a, b]$,

differences in heights on the antiderivative (such as F(b) - F(a)) correspond to the net-signed area bounded by the original function on the interval [a, b] ($\int_a^b f(x)dx$.)

For example, say that $f(x) = x^2$ and that we are interested in an antiderivative of f that satisfies $F(1) = 2$. Thinking of $a = 1$ and $b = 2$ in Equation 1, it follows from the Fundamental Theorem of Calculus that

$$F(2) = F(1) + \int_1^2 x^2 dx$$

$$= 2 + \frac{1}{3}x^3|_1^2$$

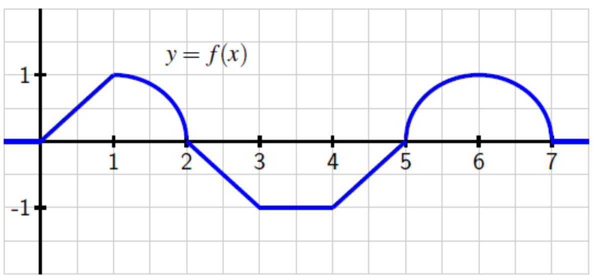
$$= 2 + (\frac{8}{3} - \frac{1}{3})$$

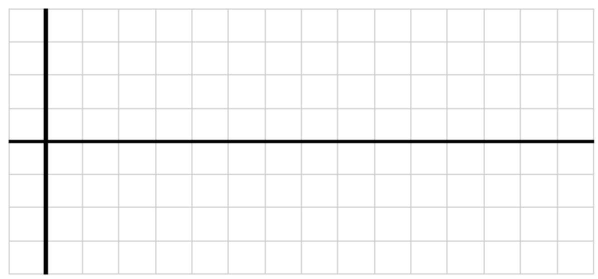
$$= \frac{13}{3}$$

In this way, we see that if we are given a function f for which we can find the exact net-signed area bounded by f on a given interval, along with one value of a corresponding antiderivative F, we can find any other value of F that we seek, and in this way construct a completely accurate graph of F. We have two main options for finding the exact net-signed area: using the Fundamental Theorem of Calculus (which requires us to find an algebraic formula for an antiderivative of the given function f), or, in the case where f has nice geometric properties, finding net-signed areas through the use of known area formulas.

Investigation 2

Suppose that the function $y = f(x)$ is given by the graph shown below, and that the pieces of f are either portions of lines or portions of circles. In addition, let F be an antiderivative of f and say that $F(0) = -1$. Finally, assume that for $x \leq 0$ and $x \geq 7$, $f(x) = 0$.





2a) On what interval(s) is F an increasing function? On what intervals is F decreasing?

b) On what interval(s) is F concave up? concave down? neither?

c) At what point(s) does F have a relative minimum? a relative maximum?

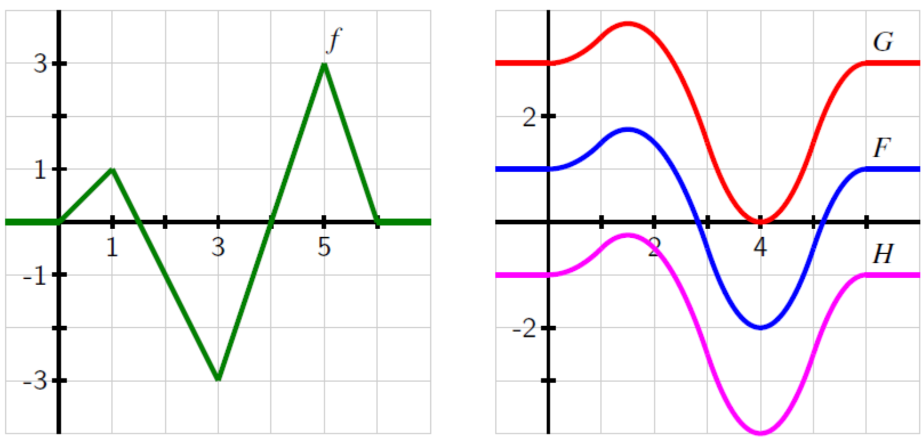
d) Use the given information to determine the exact value of $F(x)$ for $x = 1, 2, \ldots, 7$. In addition, what are the values of $F(-1)$ and $F(8)$?

e) Based on your responses to all of the preceding questions, sketch a complete and accurate graph of $y = F$ x on the axes provided, being sure to indicate the behavior of F for $x < 0$ and $x > 7$. Clearly indicate the scale on the vertical and horizontal axes of your graph.

f) What happens if we change one key piece of information: in particular, say that G is an antiderivative of f and $G(0) = 0$. How (if at all) would your answers to the preceding questions change? Sketch a graph of G on the same axes as the graph of F you constructed in (e).

III. Multiple antiderivatives of a single function

In the final question (2f above), you encountered a very important idea: a given function f has more than one antiderivative. In addition, any antiderivative of f is determined uniquely by identifying the value of the desired antiderivative at a single point. For example, suppose that f is the function shown below-left, and we say that F is an antiderivative of f that satisfies $F(0) = 1$.



At left, the graph of $y = f(x)$. At right, three different antiderivatives of f.

Then, using Equation 1, we can compute $F(1) = 1.5$, $F(2) = 1.5$, $F(3) = 0.5$, $F(4) = 2$, $F(5) = 0.5$, and $F(6) = 1$, plus we can use the fact that $F' = f$ to ascertain where F is increasing and decreasing, concave up and concave down, and has relative extremes and inflection points.

If we instead chose to consider a function G that is an antiderivative of f but has the property that $G(0) = 3$, then G will have the exact same shape as F (since both share the derivative f), but G will be shifted vertically from the graph of F, as pictured in red.

In the same way, if we assigned a different initial value to the antiderivative, say $H(0) = 1$, we would get still another antiderivative, as shown in magenta.

This example *demonstrates* that if a function has a single antiderivative, it must have infinitely many: you can add any constant to the antiderivative and get another antiderivative. For this reason, you refer to the ***general antiderivative*** of a function *f*.

For example, if $f(x) = x^2$, its general antiderivative is $F(x) = \frac{1}{3}x^3 + C$, where we include the "$+C$" to indicate that F includes all of the possible antiderivatives of f.

To identify a particular antiderivative of f, we must be provided a single value of the antiderivative F (this value is often called an ***initial condition***). In the present example, suppose that condition is $F(2) = 3$; substituting the value of 2 for x in $F(x) = \frac{1}{3}x^3 + C$, you get

$$3 = \frac{1}{3}(2)^3 + C$$

Solving for C, you get

$$C = 3 - \frac{8}{3} = \frac{1}{3}$$

Therefore, the particular antiderivative in this case is

$$F(x) = \frac{1}{3}x^3 + \frac{1}{3}$$

Investigation 3: For each of the following functions, sketch an accurate graph of the antiderivative that satisfies the given initial condition. In addition, sketch the graph of two additional antiderivatives of the given function, and state the corresponding initial conditions that each of them satisfy. If possible, find an algebraic formula for the antiderivative that satisfies the initial condition.

a) original function: $g(x) = |x| - 1$;
initial condition: $G(-1) = 0$;
interval for sketch: [-2, 2]

b) original function: $h(x) = \sin(x)$;
initial condition: $H(0) = 1$;
interval for sketch: [0, 4π]

c) original function: $\begin{cases} x^2. & \text{if } 0 \leq x \leq 1 \\ -(x-2)^2, & \text{if } 1 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$

initial condition: $P(0) = 1$;

interval for sketch: [-1, 3]

IV. Functions defined by integrals

In Equation 1, you found an important rule that enables you to compute the value of the antiderivative F at a point b, provided that you know $F(a)$ and can evaluate the definite integral from a to b of f. Again, that rule is

$$F(b) = F(a) + \int_a^b f(x)\,dx \qquad \textbf{Equation 1}$$

In several examples, you have used this formula to compute several different values of $F(b)$ and then plotted the points $(b, F(b))$ to assist us in generating an accurate graph of F. That suggests that we may want to think of b, the upper limit of integration, as a variable itself. To that end, we introduce the idea of an ***integral function***, a function whose formula involves a definite integral.

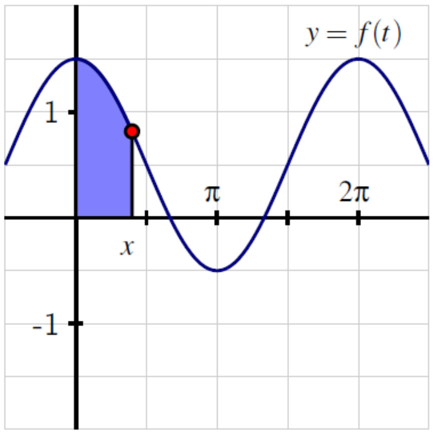
Given a continuous function f, we define the corresponding integral function A according to the rule

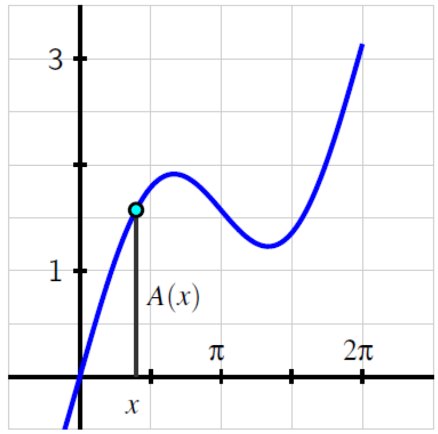
$$A(x) = \int_a^x f(t)\,dt$$

Note particularly that because you are using the variable x as the independent variable in the function A, and x determines the other endpoint of the interval over which you integrate (starting from a), you need to use a variable other than x as the variable of integration. A standard choice is t, but any variable other than x is acceptable.

One way to think of the function A is as the "net-signed area from a up to x"

function, where you consider the region bounded by $y = f(t)$ on the relevant interval. For example, in the figure below, there is a given function f pictured at left, and its corresponding area function at the right

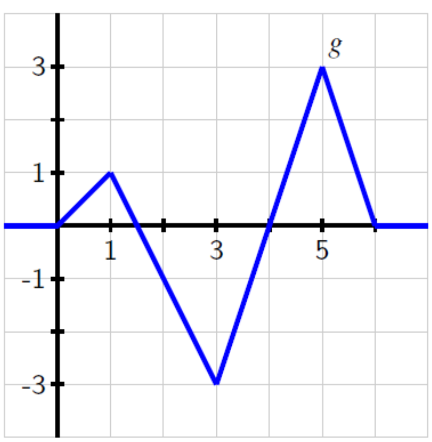


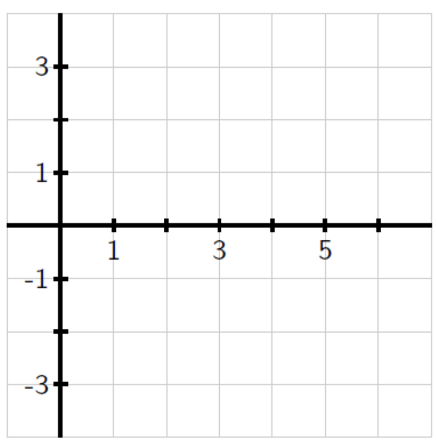


Notice that the function A measures the net-signed area from $t = 0$ to $t = x$ bounded by the curve $y = f(t)$; this value is then reported as the corresponding height on the graph of $y = A(x)$. At http://gvsu.edu/s/cz, there is a java applet that brings the static picture above to life.

The choice of a is somewhat arbitrary. In the activity that follows, we explore how the value of a affects the graph of the integral function, as well as some additional related issues.

Investigation 4: Suppose that g is given by the graph at left in Figure 5.5 and that A is the corresponding integral function defined by $A(x) = \int_1^x g(t)dt$.





a) On what interval(s) is A an increasing function? On what intervals is A decreasing? Why?

b) On what interval(s) do you think A is concave up? concave down? Why?

c) At what point(s) does A have a relative minimum? a relative maximum?

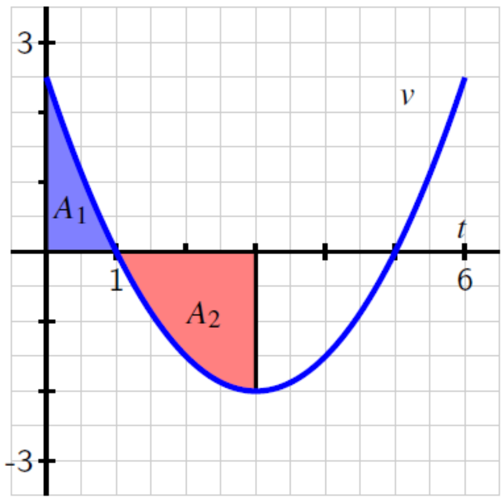
d) Use the given information to determine the exact values of $A(0)$, $A(1)$, $A(2)$, $A(3)$, $A(4)$, $A(5)$, and $A(6)$.

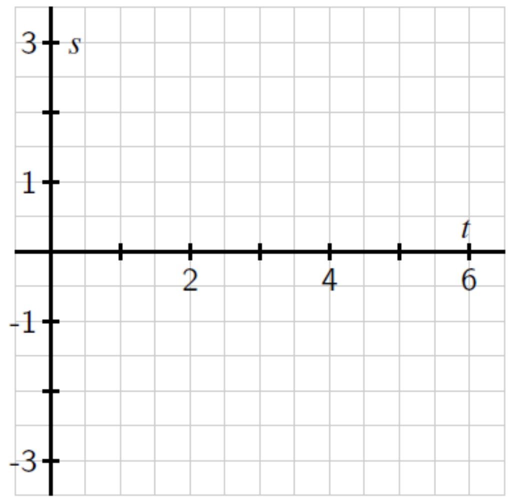
e) Based on your responses to all of the preceding questions, sketch a complete and accurate graph of $y = \mathrm{A(x)}$ on the axes provided, being sure to indicate the behavior of A for $\mathrm{x} < 0$ and $\mathrm{x} > 6$.

f) How does the graph of B compare to A if B is instead defined by $B(x) = \int_0^x g(t)dt$

V. Practice

1. A moving particle has its velocity given by the quadratic function v pictured below. In addition, it is given that $A_1 = \frac{7}{6}$ and $B_1 = \frac{8}{3}$, as well as that for the corresponding position function s, $s(0) = 0.5$.





a) Use the given information to determine $s(1)$, $s(3)$, $s(5)$, and $s(6)$.

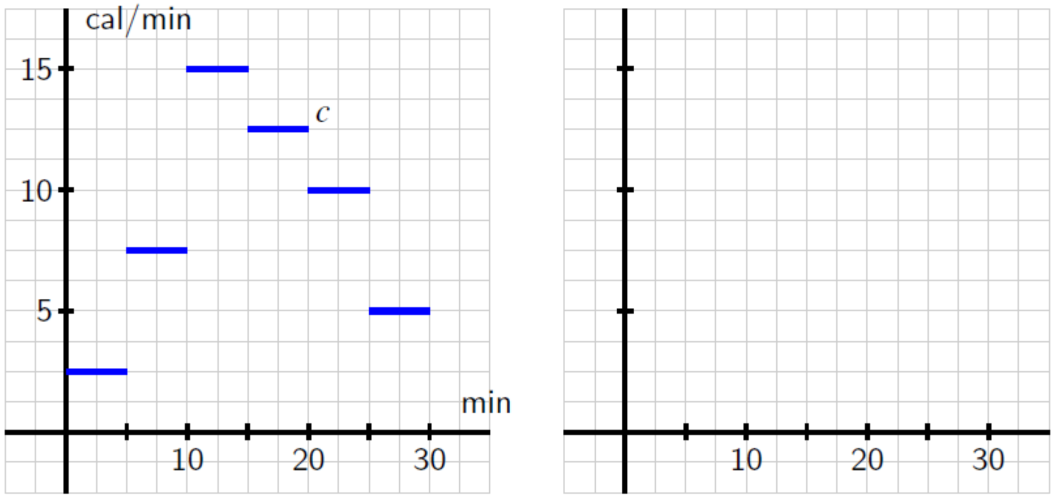
b) On what interval(s) is s increasing? On what interval(s) is s decreasing?

c) On what interval(s) is s concave up? On what interval(s) is s concave down?

d) Sketch an accurate, labeled graph of s on the axes above right.

e) Note that $v(t) = -2 + \frac{1}{2}(t-3)^2$. Find a formula for s.

2. A person exercising on a treadmill experiences different levels of resistance and thus burns calories at different rates, depending on the treadmill's setting. In one workout, the rate at which a person is burning calories is given by the piecewise constant function c pictured below. Note that the units on c are "calories per minute."



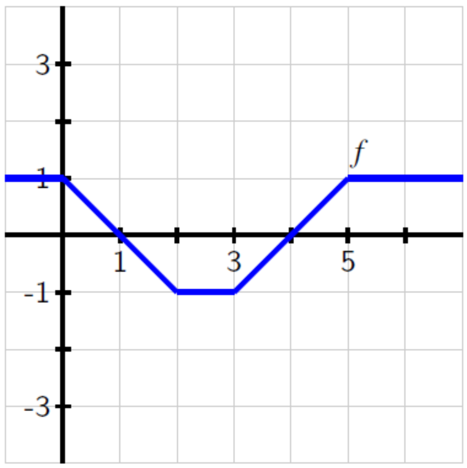
a) Let C be an antiderivative of c. What does the function C measure? What are its units?

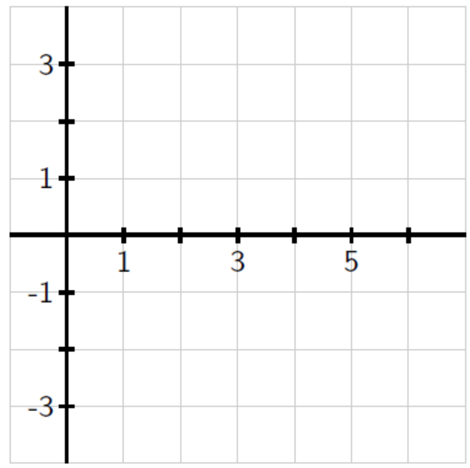
b) Assume that $C(0) = 0$. Determine the exact value of $C(t)$ at the values $t = 5$, 10, 15, 20, 25, 30.

c) Sketch an accurate graph of C on the axes provided above right. Be certain to label the scale on the vertical axis.

d) Determine a formula for C that does not involve an integral and is valid for $0 \leq t \leq 10$.

3. Consider the piecewise linear function f given below. Let the functions A, B, and C be defined by the rules $A(x) = \int_{-1}^{x} f(t)dt$, $B(x) = \int_{0}^{x} f(t)dt$, and $C(x) = \int_{1}^{x} f(t)dt$.





a) For the values x = -1, 0, 1, . . . , 6, make a table that lists corresponding values of $A(x)$, $B(x)$, and $C(x)$.

b) On the axes provided above right, sketch the graphs of A, B, and C.

c) How are the graphs of A, B, and C related?

d) How would you best describe the relationship between the function A and the function f ?

VI. Assessment – Khan Academy

1. Complete the first two online practice exercises in the *Fundamental Theorem of Calculus* unit of Khan Academy's AP Calculus AB course: https://www.khanacademy.org/math/ap-calculus-ab/fundamental-theorem-of-calculus-ab?t=practice