3.3 A – Invoking a Higher Power

The Fundamental Theorem of Calculus

In today's activities, you will turn your attention to the Biggest Idea in Big Idea 3: The Fundamental Theorem of Calculus.

Investigation 1: A student with a third-floor dormitory window 32 feet off the ground tosses a water balloon straight up in the air with an initial velocity of 16 feet per second. It turns out that the instantaneous velocity of the water balloon is given by the velocity function $v(t) = 32t + 16$, where v is measured in feet per second and t is measured in seconds.

1a) Let $s(t)$ represent the height of the water balloon above the ground at time t, and note that s is an antiderivative of v. That is, v is the derivative of s: $s'(t) = v(t)$. Find a formula for s t that satisfies the initial condition that the balloon is tossed from 32 feet above ground. In other words, make your formula for s satisfy $s(0) = 32$.

b) At what time does the water balloon reach its maximum height? At what time does the water balloon land?

c) Compute the three differences s(1) - s(0), s(2) - s(1), and s(2) - s(0). What do these differences represent?

d) What is the total vertical distance traveled by the water balloon from the time it is tossed until the time it lands?

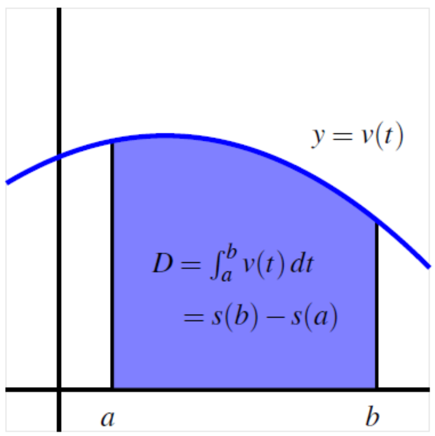
e) Sketch a graph of the velocity function $y = v(t)$ on the time interval [0, 2]. What is the total net signed area bounded by $y = v(t)$ and the t-axis on [0, 2]? Answer this question in two ways: first by using your work above, and then by using a familiar geometric formula to compute areas of certain relevant regions.

II. The Fundamental Theorem of Calculus

Consider the setting where we know the position function *s(t)* of an object moving along an axis, as well as its corresponding velocity function *v(t)*, and for the moment let us assume that *v(t)* is positive on [a, b].

Then, as shown below, you know two different perspectives on the distance, D, the object travels:

- that $D = s(b) - s(a)$ is the object's change in position
- that the distance traveled is the area under the velocity curve, which is given by the definite integral, so $D = \int_a^b v(t)\,dt$.



Of course, since both expressions tell us the distance traveled, it follows that they are equal, so

$$s(b) - s(a) = \int_a^b v(t)dt \qquad \textbf{Equation 1}$$

Understand that Equation 1 holds even when velocity is sometimes negative, since $s(b) - s(a)$ is the object's change in position over [a, b], which is simultaneously measured by the total net signed area on [a, b] given by $\int_a^b v(t)\,dt$.

Perhaps the most powerful part of Equation 1 lies in the fact that you can compute the integral's value if we can find a formula for *s*. Remember, *s* and *v* are related by the fact that *v* is the derivative of s, or equivalently that s is an ***antiderivative*** of *v*. For example, if you have an object whose velocity is $v(t) = 3t^2 + 40$ feet per second (which is always nonnegative), and wish to know the distance traveled on the interval [1, 5], then

$$D = \int_1^5 v(t)dt = \int_1^5 (3t^2 + 40)dt = s(5) - s(1)$$

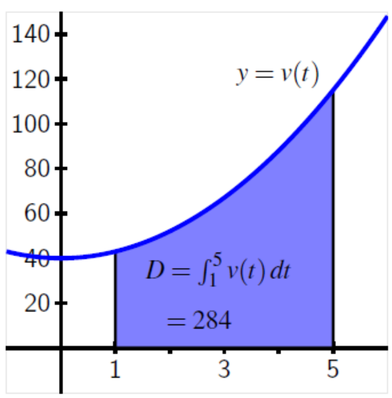
where *s* is an antiderivative of *v*. We know that the derivative of t^3 is $3t^2$ and that the derivative of 40t is 40, so it follows that if $s(t) = t^3 + 40t$, then *s* is a function whose derivative is $v(t) = s'(t) = t^3 + 40t$ v(t), and thus we have found an antiderivative of *v*.

Therefore,

$$D = \int_1^5 (3t^2 + 40)dt = s(5) - s(1)$$

$$= (5^3 + 40 \cdot 5) - (1^3 + 40 \cdot 1) = 284 \text{ feet.}$$

Note the key lesson of this example: to find the distance traveled, you need to compute the area under a curve, which is given by the definite integral. But to evaluate the integral, you found an antiderivative, *s*, of the velocity function, and then computed the total change in *s* on the interval.



Observe that you have found the exact area of the region shown in the graph, without directly computing the limit of a Riemann sum. As you proceed to thinking about contexts other than just velocity and position, it is advantageous to have a shorthand symbol for a function's antiderivative. In the general setting of a continuous function *f*, we will often denote an antiderivative of *f* by *F*, so that the relationship between F and f is that $F'(x) = f(x)$ for all relevant x. Using the notation V in place of s (so that V is an antiderivative of v) in Equation 1, it is equivalent to write that

$$\int_a^b v(t)dt = V(b) - V(a)$$

The Fundamental Theorem of Calculus (FTC[1]) summarizes these observations.

The Fundamental Theorem of Calculus: If f is a continuous function on $[a, b]$ and F is an antiderivative of f, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

A common alternate notation for $F(b)$ - $F(a)$ is

$$F(b) - F(a) = F(x)\Big|_a^b$$

where we read the right-hand side as "the function F evaluated from a to b." In this notation, the FTC says that

$$\int_a^b f(x)dx = F(x)\Big|_a^b$$

The FTC opens the door to evaluating exactly a wide range of integrals.

Example 1

$$\int_0^1 x^2 dx = \frac{1}{3}x^3\Big|_0^1$$

$$= \tfrac{1}{3}(1)^3 - \tfrac{1}{3}(0)^3$$

$$= \tfrac{1}{3}$$

[1] This abbreviation is accepted on the AP exams

Of course, finding a formula for an antiderivative can be very hard or even impossible, but you will get better with practice. You will also start recognizing patterns.

Investigation 2: Evaluating Integrals with the FTC

2. Use the Fundamental Theorem of Calculus to evaluate each of the following integrals exactly. For each, sketch a graph of the integrand on the relevant interval and write one sentence that explains the meaning of the value of the integral in terms of the (net signed) area bounded by the curve.

a) $\displaystyle\int_{-1}^{4}(2-2x)dx$

b) $\displaystyle\int_{0}^{\frac{\pi}{2}}\sin(x)dx$

c) $\displaystyle\int_{0}^{1}e^{x}dx$

d) $\displaystyle\int_{-1}^{1}x^{5}dx$

e) $\displaystyle\int_{-1}^{4}(3x^{3}-2x^{2}-e^{x})dx$

III. Basic antiderivatives

The antiderivative, or ***indefinite integral,*** of a function $f(x)$ is a function $F(x)$ whose derivative is $f(x)$.

Investigation 3: Indefinite integrals

3. Use a graphing calculator to evaluate the following – and look for patterns.

a. $\int(3x)dx$

b. $\int(5x^3)dx$

c. $\int(7x^4)dx$

d. $\int(11x^5)dx$

e. $\int(13x^6)dx$

f. (Try this without a calculator!) $\int(15x^7)dx$

4. In your own words, describe how to find an indefinite integral of a monomial. (Be precise: you have just discovered the Reverse Power Rule!)

5. Realize how easy it is to check your answer when you are trying to evaluate an indefinite integral: just differentiate (usually in your head.)

a) Take your answer from 3f above and calculate the derivative, by hand.

b) What is the derivative of $\frac{15}{8}x^8 + 3$?

c) What is the derivative of $\frac{15}{8}x^8 + 9$?

d) What is the derivative of $\frac{15}{8}x^8 + 3583$?

e) How many different possible answers could be given to question 3f?

f) How can we address this problem?

IV. The Constant of Integration

The antiderivative of any indefinite integral is not unique, that is, if $F(x)$ is an integral of $f(x)$, then so is $F(x) + C$, where C is any constant. The arbitrary constant C is called the constant of integration. The indefinite integral of $f(x)$ is wriiten as

$$\int f(x)dx = F(x) + C$$

Example 2:

$$\int x^2 dx = \frac{1}{3}x^3 + C$$

6. Correct your responses to question #3 by adding the constant of integration to each of your answer.

7. Build a list of common functions whose antiderivatives you already know, or you can intuit. (See which you can do by yourself, before asking your calculator.)

Note: don't forget your constants of integration!

Given function, $f(x)$	Antiderivative, $F(x) + C$
k (k is a constant)	
$x^n, n \neq -1$	
$\frac{1}{x}, x > 0$	
e^x	
$\sqrt{x}$ (Hint: change to a fractional power)	
sin (x)	
cos (x)	
sec (x) tan (x)	
csc (x) cot (x)	

$\sin^2(x)$	
$\csc^2(x)$	
$a^x, a > 1$	
$\dfrac{1}{1+x^2}$	
$\dfrac{1}{\sqrt{1-x^2}}$	

The Fundamental Theorem of Calculus enables you to evaluate exactly (without taking a limit of Riemann sums) any definite integral for which we are able to find an antiderivative of the integrand.

8. Use your growing knowledge of antiderivatives of basic functions to evaluate the following definite integrals. (No need for constant of integration here.)

a) $\displaystyle\int_0^2 (2x^2 - 2x + 2)dx$

a) $\displaystyle\int_0^1 (x^3 - x - e^x + 2)dx$

c) $\displaystyle\int_0^{\frac{\pi}{3}} (2\sin(t) - 4\cos(t) - \pi)dx$

d) $\displaystyle\int_0^1 \left(\sqrt{x} - x^2\right)dx$

V. The total change theorem

A slight change in notational perspective allows will allow you to gain even more insight into the meaning of the definite integral. Recall the following, where we wrote the Fundamental Theorem of Calculus for a velocity function *v* with antiderivative *V* as

$$\int_a^b v(t)dt = V(b) - V(a)$$

First, apply the Symmetric Property, and rewrite the equation as

$$V(b) - V(a) = \int_a^b v(t)dt$$

Next, replace *V* with *s* (which represents position) and replace v with *s*', (since velocity is the derivative of position), the new equation reads

$$s(b) - s(a) = \int_a^b s'^{(t)}dt$$

In words, this version of the FTC tells us that the total change in the object's position function (on a particular interval) is given by the definite integral of the position function's derivative over that interval.

Of course, this result is not limited to only the setting of position and velocity. Writing the result in terms of a more general function *f*, we have the Total Change Theorem.

The Total Change Theorem: If *f* is a continuously differentiable function on [*a*, *b*], with derivative of *f'*, then

$$f(b) - f(a) = \int_a^b f(x)dx$$

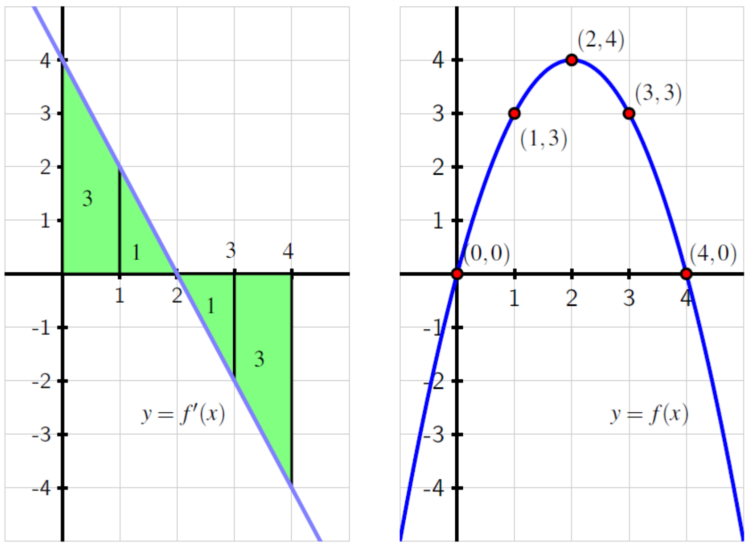
That is, the definite integral of the derivative of a function on [*a*, *b*] is the total change of the function itself on [*a*, *b*].

The Total Change Theorem tells us more about the relationship between the graph of a function and that of its derivative.

Recall the graph below, which provided one of the first times we saw that heights on the graph of the derivative function come from slopes on the graph of the function itself.

That observation occurred in the context where we knew *f* and were seeking *f'*; if now instead we think about knowing f', and seeking information about f, we can instead say the following:

Differences in heights on f correspond to net signed areas bounded by f'



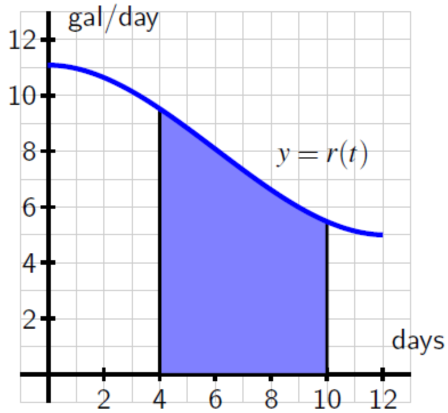
Consider the difference $f(1)$ - $f(0)$. Note that this value is 3, in part because $f(1) = 3$ and $f(0) = 0$, but also because the net signed area bounded by $y = f(x)$ on [0, 1] is 3. That is, $f(1) - f(0) = \int_0^1 f''(x)dx$. A similar pattern holds throughout, including the fact that since the total net signed area bounded by f, on [0, 4] is 0, $\int_0^4 f'(x)dx = 0$, so it must be that $f(4)$ - $f(0) = 0$, so $f(4) = f(0)$.

Beyond this general observation about area, the Total Change Theorem enables us to consider interesting and important problems where we know the rate of change, and answer key questions about the function whose rate of change we know.

Example 3: Suppose that pollutants are leaking out of an underground storage tank at a rate of $r(t)$ gallons/day, where t is measured in days. It is conjectured that $r(t)$ is given by the formula $r(t) = 0.0069t^3$? $0.125t^2 + 11.079$ over a certain 12-day period. The graph of $y = r(t)$ is given in the graph below.

a) What is the meaning of $\int_4^{10} r(t)dt$ and what is its value?

b) What is the average rate at which pollutants are leaving the tank on the time interval $4 \le t \le 10$?



Solution:

a) The definite integral $\int_4^{10} r(t)dt$ tells us the total number of gallons of pollutant that leak from the tank from day 4 to day 10.

$$\int_4^{10} r(t)dt = R(10) - R(4)$$

To compute the exact value, use FTC to find the antiderivate of $r(t)$

$$\int_4^{10} (0.0069t^3 - 0.125t^2 + 11.079)dt$$

$$= (0.0069 \cdot \frac{1}{4}t^4 - 0.125 \cdot \frac{1}{3}t^3 + 11.079t)\Big|_4^{10}$$

$$\approx 44.282$$

To figure out the new units: $\frac{\text{gallons}}{\cancel{\text{day}}} \cdot \cancel{\text{days}} = \text{gallons}$

Thus, approximately 44.282 gallons of pollutant leaked over the six-day time period.

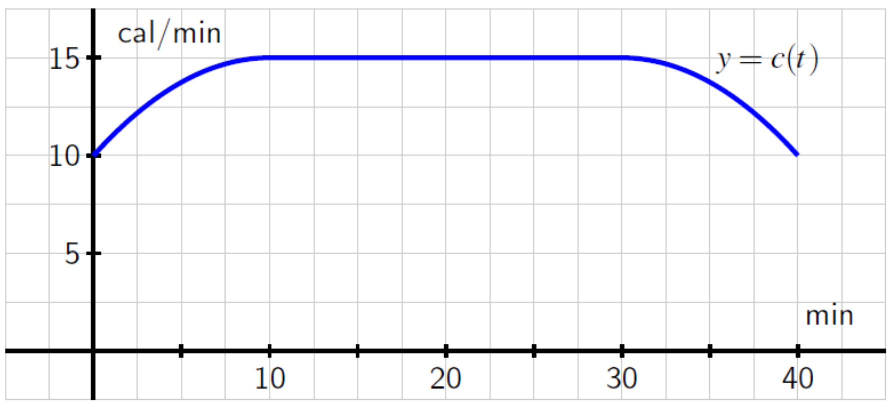
b) To find the average rate at which pollutant leaked from the tank over $4 \leq t \leq 10$, compute the average value of r on [4, 10]. Thus,

$$r_{\text{AVG}[4,10]} = \frac{1}{10-4}\int_4^{10} r(t)dt$$

$$\approx \frac{44.282}{6} \approx 7.380 \text{ gallons per day.}$$

VI. Practice

1. During a 40-minute workout, a person riding an exercise machine burns calories at a rate of c calories per minute, where the function $y = c(t)$ is shown below. On the interval $0 \le t \le 10$, the formula for c is $c(t) = -0.05t^2 + t + 10$, while on $30 \le t \le 40$, its formula is $c(t) = -0.05t^2 + 3t - 30$.



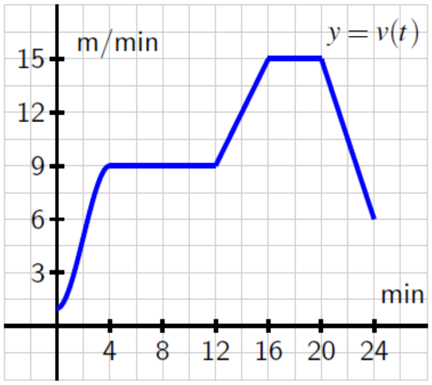
a) What is the total number of calories the person burns during the first 10 minutes of her workout?

b) Let $C(t)$ be an antiderivative of $c(t)$. What is the meaning of $C(40) - C(0)$ in the context of the person exercising? Include units in your answer.

c) Determine the exact average rate at which the person burned calories during the 40-minute workout.

d) At what time(s), if any, is the instantaneous rate at which the person is burning calories equal to the average rate at which she burns calories, on the time interval $0 \le t \le 40$?

2. The instantaneous velocity (in meters per minute) of a moving object is given by the function v as pictured in Figure 4.37. Assume that on the interval $0 \le t \le 4$, v(t) is given by $v(t) = -\frac{1}{4}t^3 + \frac{3}{2}t^2 + 1$, and that on every other interval v is piecewise linear, as shown below.



a) Determine the total distance traveled by the object on the time interval $0 \le t \le 4$.

b) What is the object's average velocity on [12, 24]?

c) At what time is the object's acceleration greatest?

d) Suppose that the velocity of the object is increased by a constant value c for all values of t. What value of c will make the object's total distance traveled on [12, 24] be 210 meters?

3. A function f is given piecewise by the formula

$$f(x) = \begin{cases} -x^2 + 2x + 1, & \text{if } 0 \leq x \leq 2 \\ -x + 3, & \text{if } 0 \leq x \leq 2 \\ x^2 - 8x + 15, & \text{if } 0 \leq x \leq 2 \end{cases}$$

a) Determine the exact value of the net signed area enclosed by f and the x-axis on the interval [2, 5].

b) Compute the exact average value of f on [0, 5].

c) Find a formula for a function g on $5 \leq x \leq 7$ so that if we extend the above definition of f so that $f(x) = g(x)$ on $5 \leq x \leq 7$, it follows that $\int_0^7 f(x)dx = 0$.

4. When an aircraft attempts to climb as rapidly as possible, its climb rate (in feet per minute) decreases as altitude increases, because the air is less dense at higher altitudes. Given below is a table showing performance data for a certain single engine aircraft, giving its climb rate at various altitudes, where $c(h)$ denotes the climb rate of the airplane at an altitude h.

h (feet)	0	1000	2000	3000	4000	5000	6000	7000	8000	9000	10000
c ($\frac{\text{ft}}{\text{min}}$)	925	875	830	780	730	685	635	585	535	490	440

Let a new function called $m(h)$ measure the number of minutes required for a plane at altitude h to climb the next foot of altitude.

a) Determine a similar table of values for $m(h)$ and explain how it is related to the table above. Be sure to explain the units.

b) Give a careful interpretation of a function whose derivative is $m(h)$. Describe what the input is and what the output is. Also, explain in plain English what the function tells us.

c) Determine a definite integral whose value tells us exactly the number of minutes required for the airplane to ascend to 10,000 feet of altitude. Clearly explain why the value of this integral has the required meaning.

d) Use the Riemann sum M_5 to estimate the value of the integral you found in (c). Include units on your result.

5. In Unit 1, you learned that for an object moving along a straight line with position function $s(t)$, the object's "average velocity on the interval $[a, b]$" is given by

$$AV_{[a,b]} = \frac{s(b) - s(a)}{b - a}$$

In this lesson, you saw that for an object moving along a straight line with velocity function $v(t)$, the object's "average value of its velocity function on $[a, b]$"

$$V_{\text{AVG}[a,b]} = \frac{1}{b - a} \int_a^b v(t)dt$$

Are the "average velocity on the interval $[a, b]$" and the "average value of the velocity function on $[a, b]$" the same thing? Why or why not? Explain.

VII. Assessment – Khan Academy

1. Complete the first eight online practice exercises in the Indefinite Integral unit of Khan Academy's AP Calculus AB course: https://www.khanacademy.org/math/ap-calculus-ab/indefinite-integrals-ab?t=practice

2. Optional: Practice 9 Challenge