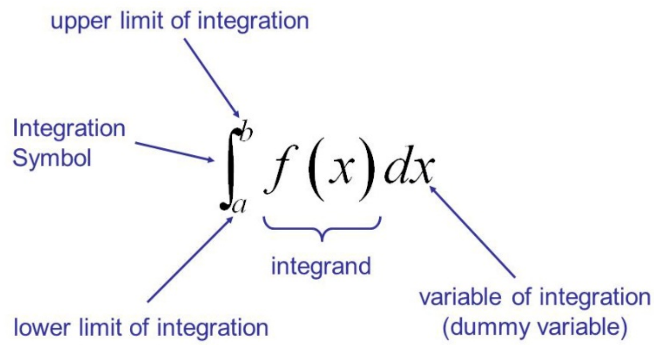
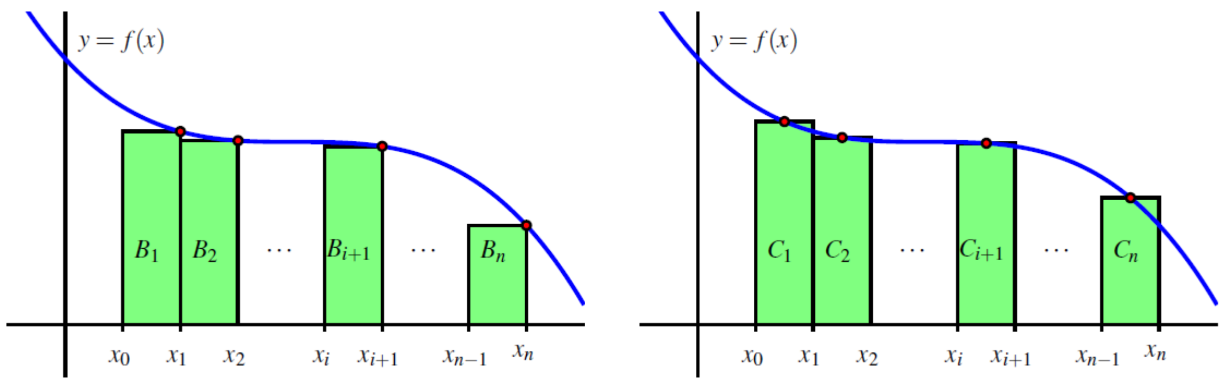
3.2 A2 - Just Like Derivatives…but Backwards



The Definite Integral

In the previous lesson, you saw that as the number of rectangles got larger and larger, the values of L_n, M_n, and R_n all grew closer and closer to the same value. It turns out that this occurs for any continuous function on an interval [a, b], and even more generally for a Riemann sum using any point x_{i+1} in the interval $[x_i, x_{i+1}]$.



Said differently, as we let $n \to \infty$, it doesn't really matter where we choose to evaluate the function within in a given subinterval, because

$$\lim_{n\to\infty} L_n = \lim_{n\to\infty} R_n = \lim_{n\to\infty} M_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i) \cdot \Delta x$$

That these limits always exist (and share the same value) for a continuous function f allows us to make the following definition:

Definition: The ***definite integral*** of a continuous function f on the interval [a, b], denoted $\int_a^b f(x)\,dx$, is the real number given by

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i) \cdot \Delta x$$

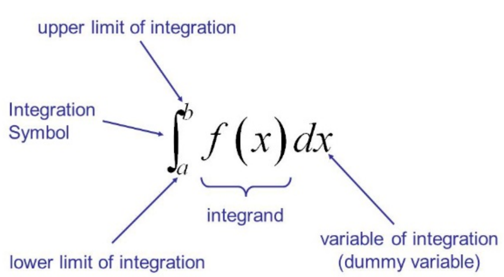
1. Explain the following terms, as used in the definition of an integral. Refer to the graph on the previous page, if needed.

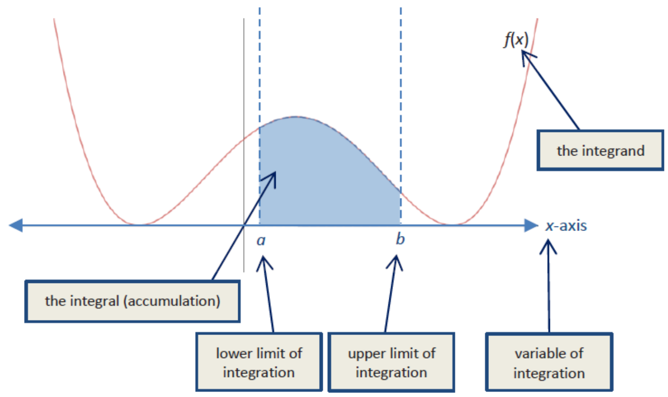
a) $\lim\limits_{n \to \infty} \sum\limits_{i=1}^{n}$

b) $f(x_i)$

c) $\triangle \mathrm{x}$

d) $\int_a^b f(x)\,dx$





We call the symbol the ***integral sign***, the values a and b the ***limits of integration***, and the function f the ***integrand***. The process of determining the real number $\int_a^b f(x)\,dx$ is called ***evaluating the definite integral***. While there are several different interpretations of the value of the definite integral, for now the most important is that $\int_a^b f(x)\,dx$ measures the net signed area bounded by $y = f(x)$ and the x-axis on the interval [a, b].

Investigation 1: TI-Nspire Investigation – Definite Integral

- Open the Definite Integral document on a TI Nspire handheld or computer app
  - Alternate site
- Complete the Student Activity worksheet
  - Alternate site
- After completing the investigation, answer questions 2-8 below.

2. Calculate the following definite integrals using a calculator or other precise means

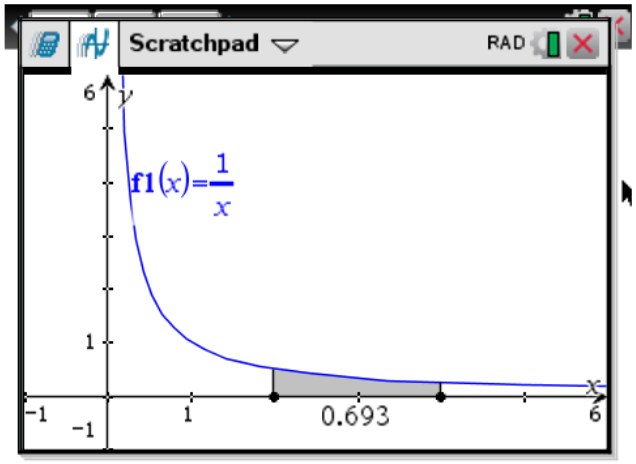
a) $\displaystyle\int_{-1}^{1} x^2 - x - 1 \; dx$

b) $\displaystyle\int_{1}^{2} \frac{3x-1}{3x} dx$

c) $\displaystyle\int_{-1}^{0} \sqrt{3u+4}\, du$

d) $\displaystyle\int_{0}^{\frac{\pi}{6}} \frac{\cos\theta}{1+2\sin\theta} d\theta$

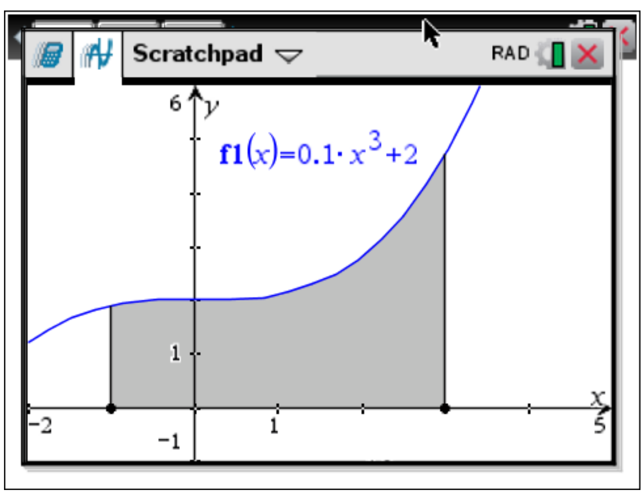
e) $\displaystyle\int_{0}^{1} \frac{e^x}{e^x+1} dx$



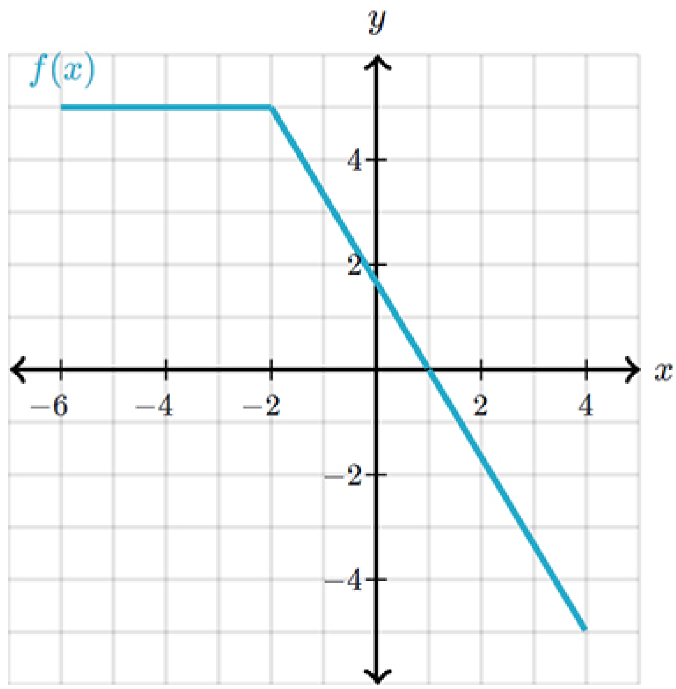
3. Which of the following integrals represent the area under the curve shown above?

a) $\int_1^3 \frac{1}{x}\,dx$ b) $\int_2^4 \frac{1}{x}\,dx$ c) $\int_3^6 \frac{1}{x}\,dx$

4. Does switching the order of the upper and lower limits of integration affect the calculation. In the screen shot above 0.693 units squared represents the area. Calculate this integral and then reverse the order of the limits. Describe the results. Why does this happen?



5. Write an integral that represents the area of the shaded region in the graph above.



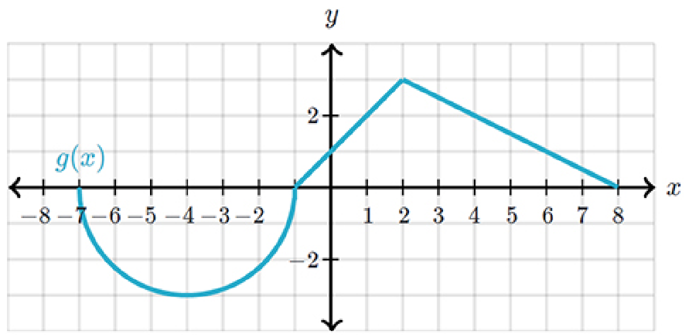
6. Calculate the exact area under the graph of $f(x)$ represented by the definite integrals below. (Hint: use the graph and your knowledge of area, not a calculator!)

a) $\int_{-6}^{1} f(x)\,dx =$

b) $\int_{1}^{4} f(x)\,dx =$

Did you remember that some of the area in 5b is negative?

c) $\int_{1}^{4} f(x)\,dx =$

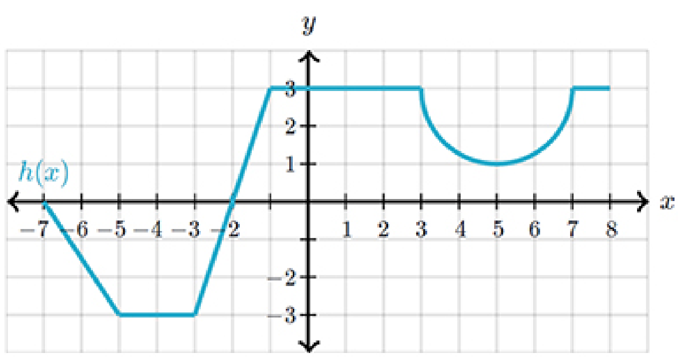


7. Calculate the exact area under the graph of $g(x)$ represented by the definite integrals below. (Leave answers in terms of π.)

a) $\int_{-7}^{-1} g(x)\,dx =$

b) $\int_{-1}^{8} g(x)\,dx =$

c) $\int_{-7}^{8} g(x)\,dx =$



8. Calculate the exact area under the graph of $h(x)$ represented by the definite integrals below. (Leave answers in terms of π.)

a) $\displaystyle\int_{-7}^{-2} h(x)\,dx =$

b) $\displaystyle\int_{-5}^{1} h(x)\,dx =$

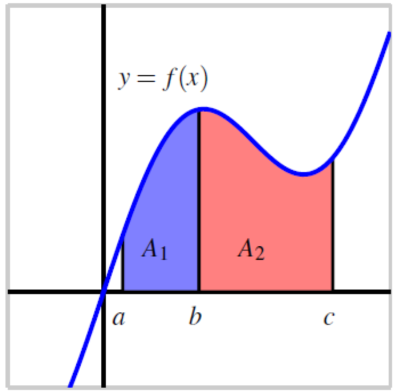
c) $\displaystyle\int_{-1}^{8} h(x)\,dx =$

II. Properties of the definite integral

With the perspective that the definite integral of a function f over an interval [a, b] measures the net signed area bounded by f and the x-axis over the interval, there are some standard properties of the definite integral.

Single Point Integral: If you consider the definite integral $\int_a^a f(x)\,dx$ for any real number a, it is evident that no area is being bounded because the interval begins and ends with the same point. Hence,

> If f is a continuous function and a is a real number, then
> $$\int_a^a f(x)\,dx = 0$$



Subdividing a Definite Integral: In the graph above, it should be evident that the area of the entire shaded region is equal to $A_1 + A_2$. In other words,

$$\int_a^b f(x)dx = A_1, \int_b^c f(x)dx = B_1, and \int_a^c f(x)dx = A_1 + A_2$$

which is indicative of the following general rule.

If f is a continuous function and a, b and c are real numbers, then

$$\int_a^c f(x)\,dx = \int_a^b f(x)dx + \int_b^c f(x)dx$$

.

Of course, you can break any definite interval into three or more pieces. While this rule is most apparent in the situation where $a < b < c$, it in fact holds in general for any values of a, b, and c.

This result is connected to another property of the definite integral, which states that if we **reverse the order of the limits of integration**, we change the sign of the integral's value.

If f is a continuous function and a and b are real numbers, then

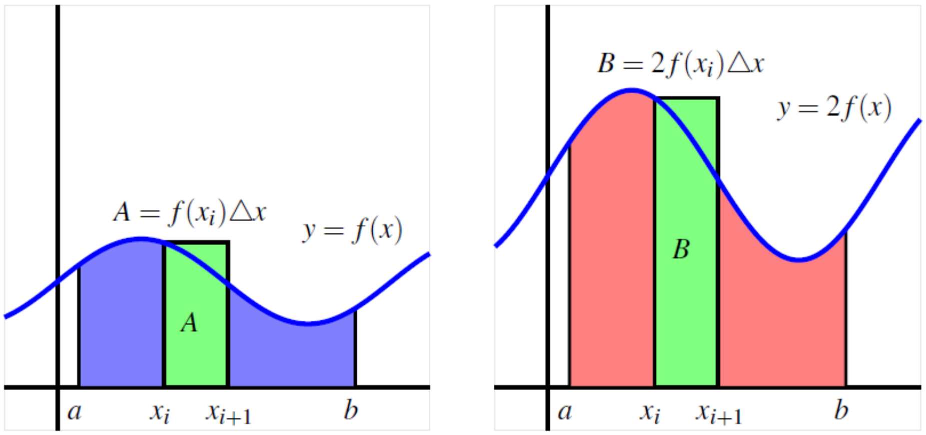
$$\int_a^b f(x)\,dx = -\int_b^a f(x)dx$$

.

Note: You should have observed this property in question 4.

There are two additional properties of the definite integral that you need to understand. Recall that when you worked with derivative rules in Big Idea 2, you found that both the Constant Multiple Rule and the Sum Rule held.

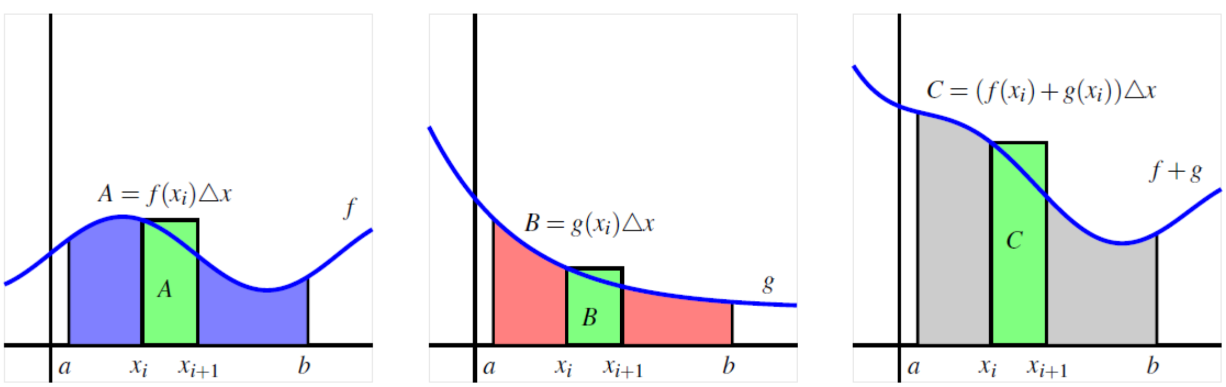
It turns out that similar rules hold for the definite integral.



Constant Multiple Rule: If f is a continuous function and k is a real number, then

$$\int_a^b k \cdot f(x)\, dx \;=\; k\int_a^b f(x)dx$$

Finally, there is a similar situation geometrically with the sum of two functions f and g. As seen below, if you take the sum of two functions f and g, at every point in the interval, the height of the function $f + g$ is given by $(f+g)(x_i) = f(x_i) + g(x_i)$, which is the sum of the individual function values of f and g (taken at left endpoints).



Hence, for the pictured rectangles with areas A, B, and C, it follows that $C = A + B$, and because this will occur for every such rectangle, in the limit the area of the gray region will be the sum of the areas of the blue and red regions. Stated in terms of definite integrals, please note the following general rule.

Sum Rule: If f and g are continuous functions, then

$$\int_a^b [f(x) + g(x)]\, dx \ = \int_a^b f(x)dx + \int_a^b g(x)\, dx$$

.

More generally, the Constant Multiple and Sum Rules can be combined so that for any continuous functions f and g and any constants c and k,

$$\int_a^b [cf(x) \pm kg(x)]\, dx \ = c\int_a^b f(x)dx \pm k\int_a^b g(x)\, dx$$

Investigation 2: Properties of Definite Integrals

9. Suppose that the following information is known about the functions f, g, x^2, and x^3:

- $\int_0^2 f(x)dx = -3$; $\int_2^5 f(x)dx = 2$
- $\int_0^2 g(x)dx = 4$; $\int_2^5 g(x)dx = -1$
- $\int_0^2 x^2 dx = \frac{8}{3}$; $\int_2^5 x^2 dx = \frac{117}{3}$
- $\int_0^2 x^3 dx = 4$; $\int_2^5 x^3 dx = \frac{609}{4}$

Use the provided information and the rules discussed in the preceding section to evaluate each of the following definite integrals.

a) $\int_5^2 f(x)dx =$

b) $\int_0^5 g(x)dx =$

c) $\int_0^5 [f(x) + g(x)]dxdx =$

d) $\int_2^5 (3x^2 - 4x^3)dx =$

e) $\int_5^0 (2x^3 - 7g(x))dx =$

f) $\int_2^2 g(x)dx =$

III. A function's average value

One of the most valuable applications of the definite integral is that it provides a way to meaningfully discuss the ***average value of a function***, even for a function that takes on infinitely many values. Recall that if we wish to take the average of n numbers $y_1, y_2, \ldots, y_n$, we do so by computing

$$\text{Avg} = \frac{y_1+y_2+\cdots+y_n}{n}$$

Since integrals arise from Riemann sums in which we add n values of a function, it should not be surprising that evaluating an integral is something like averaging the output values of a function. Consider, for instance, the right Riemann sum R_n of a function f, which is given by

$$R_n = f(x_1)\cdot\Delta x + f(x_2)\cdot\Delta x + \cdots f(x_n)\cdot\Delta x = (f(x_1) + (f(x_2) + \cdots \big(f(x_n)\big)\,\Delta x$$

Since $\Delta x = \frac{b-a}{n}$, we can thus write

$$\begin{aligned} R_n &= (f(x_1) + (f(x_2) + \cdots + \big(f(x_n)\big)\cdot\frac{b-a}{n} \\ &= (b-a)\frac{(f(x_1) + (f(x_2) + \cdots (f(x_n)}{n} \qquad \text{(Equation 1)} \end{aligned}$$

Here, we see that the right Riemann sum with n subintervals is the length of the interval (b – a) times the average of the n function values found at the right endpoints. Notice that the larger the value of n you use, the more accurate your average of the values of f will be. Indeed, we will define the average value of f on [a, b] to be

$$f_{\text{AVG}[a,b]} = \lim_{n\to\infty} \frac{(f(x_1) + (f(x_2) + \cdots (f(x_n)}{n}$$

But remember that for any continuous function f on [a, b], taking the limit of a Riemann sum leads precisely to the definite integral. That is, $\lim_{n\to\infty} R_n = \int_a^b f(x)dx$ and therefore taking the limit as $n \to \infty$ in Equation 1, you now have

$$\int_a^b f(x)dx = (b-a) \cdot f_{\mathrm{AVG}[a,b]} \qquad \text{(Equation 2)}$$

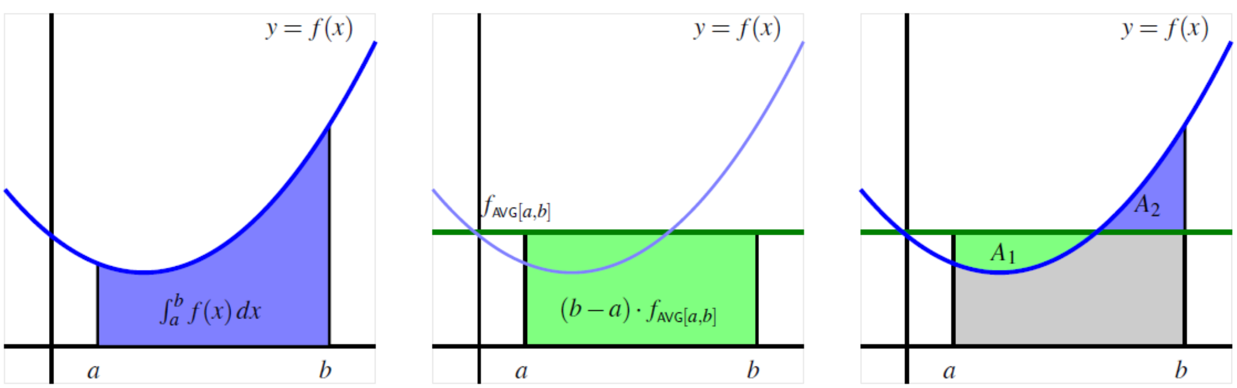
Solving this equation for $f_{\mathrm{AVG}[a,b]}$, results in the following general principle.

Average value of a function: If f is a continuous function on [a, b], then its average value on [a, b] is given by the formula

Average Value of a function: If f is a continuous function on [a, b], then its average value on [a, b] is given by the formula

$$f_{\mathrm{AVG}[a,b]} = \frac{1}{b-a} \cdot \int_a^b f(x)dx$$

Observe that Equation 2 tells us another way to interpret the definite integral: the definite integral of a function f from a to b is the length of the interval (b - a) times the average value of the function on the interval. In addition, Equation 2 has a natural visual interpretation when the function f is nonnegative on [a, b].



Notice above where you see at left the shaded region whose area is $\int_a^b f(x)dx$, at center the shaded rectangle whose dimensions are $(b-a)$ by $f_{\mathrm{AVG}[a,b]}$, and at right these two figures superimposed.

Specifically, note that in dark green you see the horizontal line $y = f_{\text{AVG}[a,b]}$. Thus, the area of the green rectangle is given by $(b - a) \cdot f_{\text{AVG}[a,b]}$, which is precisely the value of $\int_a^b f(x)dx$. Said differently, the area of the blue region in the left figure is the same as that of the green rectangle in the center figure; this can also be seen by observing that the areas A_1 and A_2 in the rightmost figure appear to be equal.

Ultimately, the average value of a function enables us to construct a rectangle whose area is the same as the value of the definite integral of the function on the interval.

Investigation 3: Average Value

10. Suppose that $v(t) = \sqrt{4 - (t-2)^2}$ tells us the instantaneous velocity of a moving object on the interval $0 \le t \le 4$, where t is measured in minutes and v is measured in meters per minute.

a) Sketch an accurate graph of $y = v(t)$. What kind of curve is $y = \sqrt{4 - (t-2)^2}$?

b) Evaluate $\int_0^4 v(t)dt$ exactly.

c) In terms of the physical problem of the moving object with velocity v(t), what is the meaning of $\int_0^4 v(t)dt$? Include units in your response.

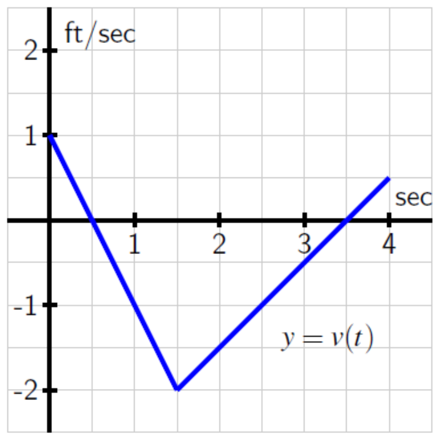
d) Determine the exact average value of $v(t)$ on [0, 4]. Include units in your answer.

e) Sketch a rectangle whose base is the line segment from $t = 0$ to $t = 4$ on the t-axis such that the rectangle's area is equal to the value of $\int_0^4 v(t)dt$. What is the rectangle's exact height?

(f) How can you use the average value you found in (d) to compute the total distance traveled by the moving object over [0, 4]?

IV. Exercises

1. The velocity of an object moving along an axis is given by the piecewise linear function v that is pictured below. Assume that the object is moving to the right when its velocity is positive, and moving to the left when its velocity is negative. Assume that the given velocity function is valid for t = 0 to t = 4.



a) Write an expression involving definite integrals whose value is the total change in position of the object on the interval [0, 4].

b) Use the provided graph of v to determine the value of the total change in position on [0, 4].

c) Write an expression involving definite integrals whose value is the total distance traveled by the object on [0, 4]. What is the exact value of the total distance traveled on [0, 4]?

d) What is the object's exact average velocity on [0, 4]?

e) Find an algebraic formula for the object's position function on [0, 1.5] that satisfies $s(0) = 0$.

2. Suppose that the velocity of a moving object is given by $v(t) = t(t-1)(t-3)$, measured in feet per second, and that this function is valid for $0 \leq t \leq 4$.

a) Write an expression involving definite integrals whose value is the total change in position of the object on the interval [0, 4].

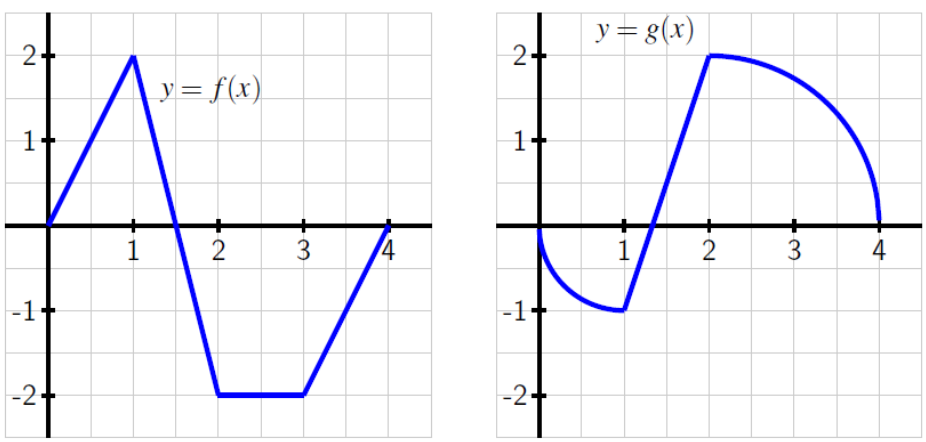
b) Use appropriate technology (on most graphing calculators) to compute to estimate the object's total change in position on [0, 4]. Work to ensure that your estimate is accurate to three decimal places, and explain how you know this to be the case.

c) Write an expression involving definite integrals whose value is the total distance traveled by the object on [0, 4].

d) Use appropriate technology to estimate the object's total distance travelled on [0, 4]. Work to ensure that your estimate is accurate to three decimal places, and explain how you know this to be the case.

e) What is the object's average velocity on [0, 4], accurate to three decimal places?

3. Consider the graphs of two functions f and g that are provided in Figure 4.30. Each piece of f and g is either part of a straight line or part of a circle.



a) Determine the exact value of $\int_0^1 [f(x) + g(x)]dx$.

b) Determine the exact value of $\int_1^4 [2f(x) - 3g(x)]dx$.

c) Find the exact average value of $h(x) = g(x) - f(x)$ on [0, 4].

d) For what constant c does the following equation hold?

$$\int_0^4 c\,dx = \int_0^4 [f(x) + g(x)]dx$$

4. Let $f(x) = 3 - x^2$ and $g(x) = 2x^2$

a) On the interval [-1, 1], sketch a labeled graph of $y = f(x)$ and write a definite integral whose value is the exact area bounded by $y = f(x)$ on [-1, 1].

b) On the interval [-1, 1], sketch a labeled graph of $y = g(x)$and write a definite integral whose value is the exact area bounded by y = g(x) on [-1, 1].

c) Write an expression involving a difference of definite integrals whose value is the exact area that lies between $y = f(x)$ and $y = g(x)$ on [-1, 1].

d) Explain why your expression in (c) has the same value as the single integral

$$\int_{-1}^{1} [f(x) - g(x)]dx$$

e) Explain why, in general, if $p(x) \geq q(x)$ for all x in [a, b], the exact area between $y = p(x)$ and $y = q(x)$ is given by

$$\int_a^b [p(x) - q(x)]dx$$

V. Assessment – Khan Academy

1. Complete the first four online practice exercises in the *Definite Integrals Introduction* unit of Khan Academy's AP Calculus AB course: https://www.khanacademy.org/math/ap-calculus-ab/definite-integrals-intro-ab?t=practice

2. And from previous unit: https://www.khanacademy.org/math/ap-calculus-ab/riemann-sums-ab/riemann-sums-with-sigma-notation-ab/e/riemann-sums-and-sigma-notation