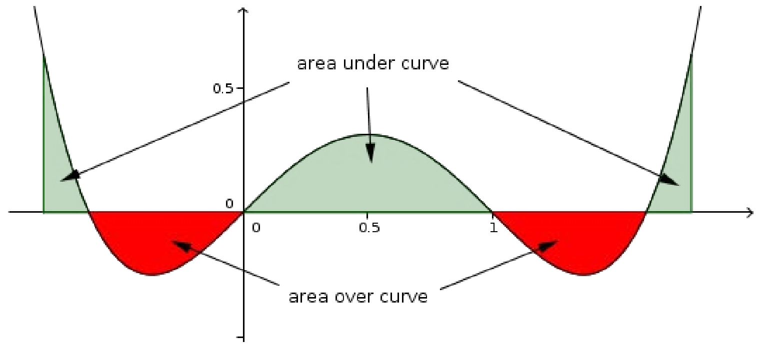
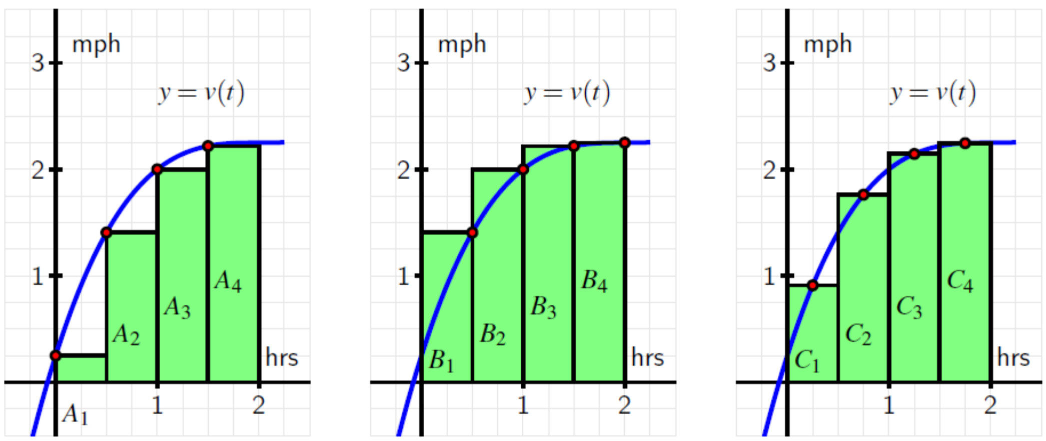
3.2A1 - Riemann Sums



Estimating the Area Under/Over a Curve

Investigation 3.2.1: A person walking along a straight path has her velocity in miles per hour at time t given by the function $v(t) = 0.25t^3\ 1.5t^2 + 3t + 0.25$, for times in the interval $0 \leq t \leq 2$. The graph of this function is also given in each of the three diagrams below. Note that in each diagram, four rectangles are used to estimate the area under $y = v(t)$ on the interval $0 \leq t \leq 2$, but the method by which the four rectangles' respective heights are decided varies among the three individual graphs.



1. How are the heights of rectangles in the left-most diagram being chosen? Explain, and hence determine the value of

$$S = A_1 + A_2 + A_3 + A_4$$

by evaluating the function $y = v(t)$ at appropriately chosen values and observing the width of each rectangle.

2. Explain how the heights of rectangles are being chosen in the middle diagram and find the value of

$$T = B_1 + B_2 + B_3 + B_4.$$

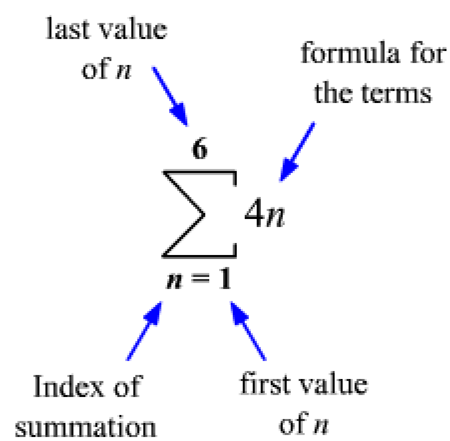
3. Likewise, Explain how the heights of rectangles are chosen in the right-most diagram and determine

$$U = C_1 + C_2 + C_3 + C_4.$$

4. Of the estimates S, T, and U, which do you think is the best approximation of D, the total distance the person traveled on [0, 2]? Why?

II. Sigma Notation

It is apparent from several different problems you have considered that sums of areas of rectangles is one of the main ways to approximate the area under a curve over a given interval. Intuitively, we expect that using a larger number of thinner rectangles will provide a way to improve the estimates we are computing. As such, we anticipate dealing with sums with many terms. To do so, we introduce the use of so-called ***sigma notation***, named for the Greek letter Σ, which is the capital letter S in the Greek alphabet.



For example, say we are interested in the sum

$1 + 2 + 3 + \cdots + 100,$

which is the sum of the first 100 natural numbers. Sigma notation provides a shorthand notation that recognizes the general pattern in the terms of the sum. It is equivalent to write

$$\sum_{k=1}^{100} k$$

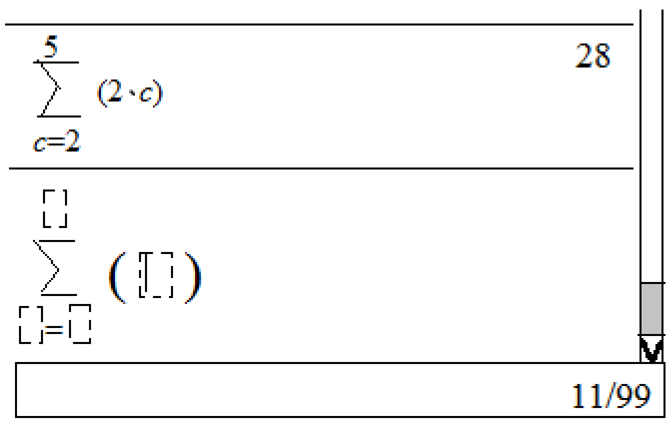
We read the symbol

$$\sum_{k=1}^{100} k$$

as "the sum from k equals 1 to 100 of k." The variable k is k=1 usually called the index of summation, and any letter can be used for the index. Each sum in sigma notation involves a function of the index; for example,

$$\sum_{k=1}^{10} (k^2 + 2k) = (1^2 + 2 \cdot 1) + (2^2 + 2 \cdot 2) + (3^2 + 2 \cdot 3) + \cdots + (10^2 + 2 \cdot 10)$$

Most calculators with newer operating systems have a summation function, which can by reached by pressing Menu > Calculus > Sum (nSpire) or Math > 0 (TI-84).



Screen shot from a TI-Nspire

Investigation 3.2.2: For each sum written in sigma notation, write the sum long-hand and evaluate the sum to find its value. For each sum written in expanded form, write the sum in sigma notation.

5(a) $\sum_{k=1}^{5} (k^2 + 2)$

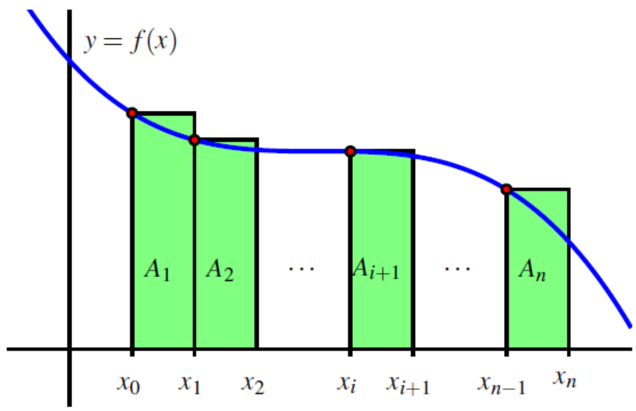
5(b) $\sum_{i=1}^{6} (2i - 1)$

5(c) $3 + 7 + 11 + 15 + \ldots + 27$

5(d) $3 + 7 + 11 + 15 + \ldots + 27$

5(e) $\sum_{i=1}^{5} \frac{1}{2^i}$

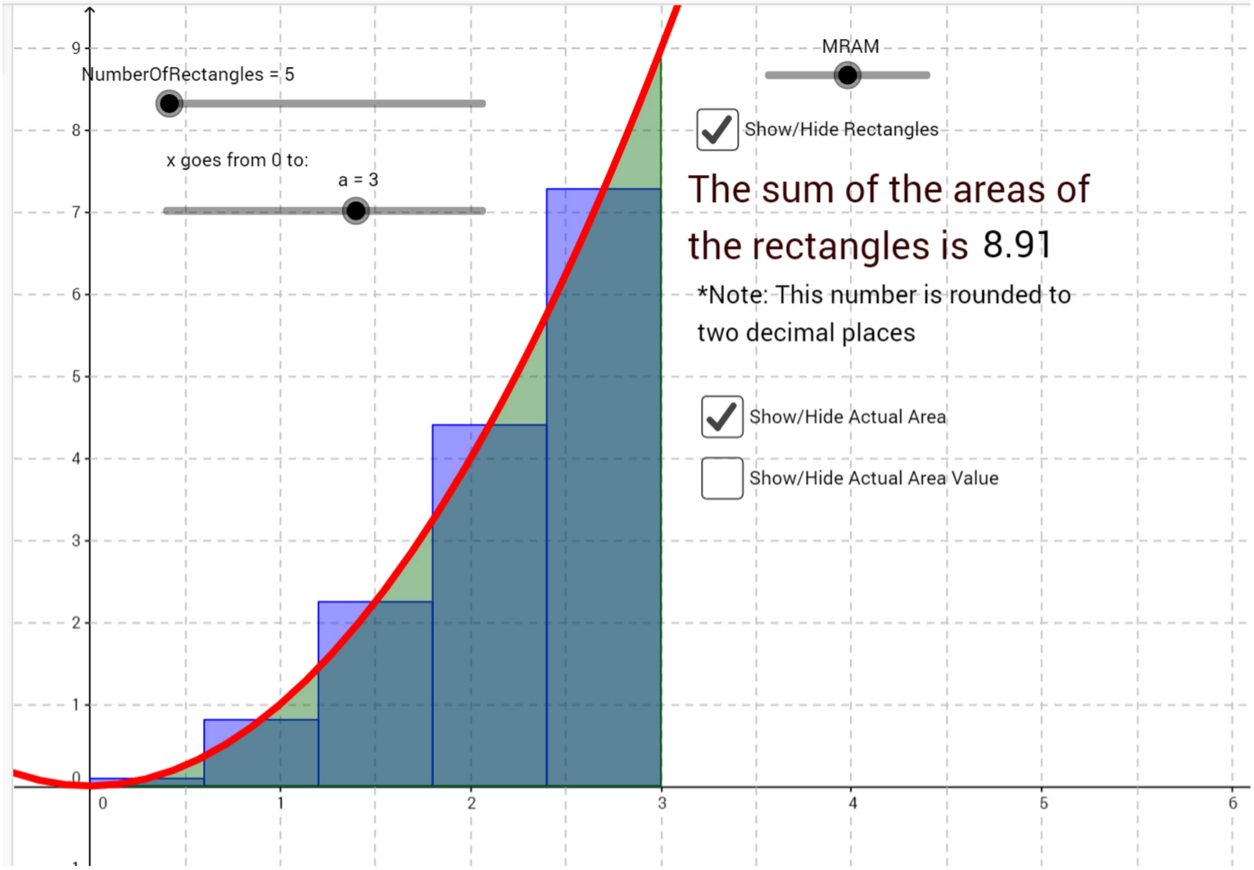
III. Riemann Sums



A ***Riemann sum*** is a method to approximate the area under a curve. It is named after German mathematician [Bernhard Riemann].

6. Write, in sigma notation, a formula for the area under $y = f(x)$ represented by the yellow rectangles above.

7. What would make this method of approximation more accurate?



Investigation 3.2.3:

Explore the GeoGebra applet at https://ggbm.at/da3XKZB5 and answer the following questions. (The recycle symbol in the upper right corner rests the app.)

8. Use the “Number of Rectangles” slider in the upper right. What is the estimation of the area under the curve $y = x^2$ over the interval from $0 \leq x \leq 3$ when it is broken into 6 rectangles? Into 11 rectangles? Into 18 rectangles?

9. What limit does the area appear to be approaching?

10. Reset the number of rectangles to 11, and adjust the upper right slider. Compare the three settings of interest: LRAM, MRAM and RRAM. These are three different methods of making a Riemann sum approximation. In your own words, describe the different ways of

finding the height of the rectangles.

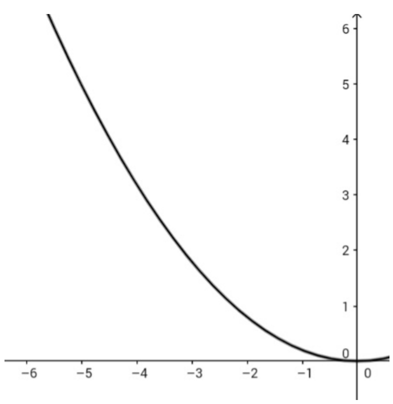
a) Left Riemann Sum (LRAM or L_n):

a) Midpoint Riemann Sum (MRAM or M_n):

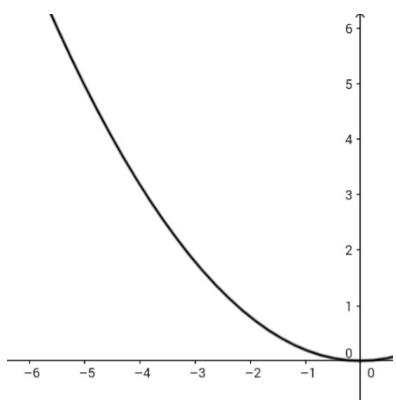
a) Right Riemann Sum (RRAM or R_n):

11. Which of the methods appears to be most accurate? Why? Which of the other methods overestimates the area and which one underestimates the area under $y = x^2$ from [0 , 3]?

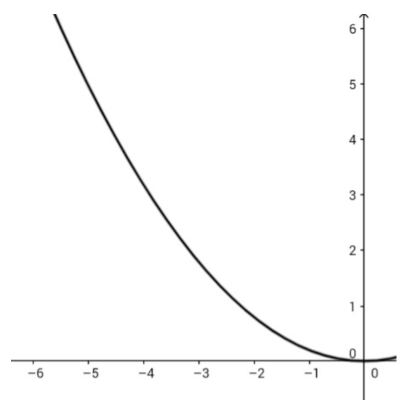
12. Sketch a R_5, R_5 and R_5 for the three second quadrant graphs of $y = .2x^2$ from [-5 , 0] shown below. Divide each region into five equal subintervals.



L_5



M_5

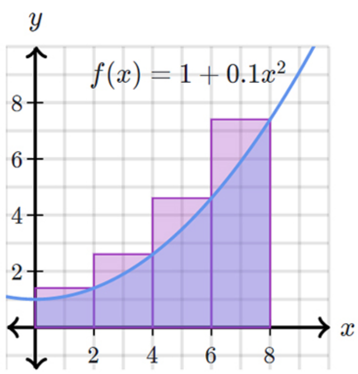


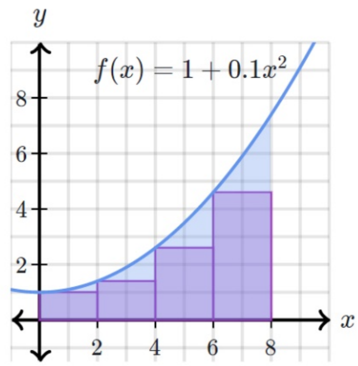
R_5

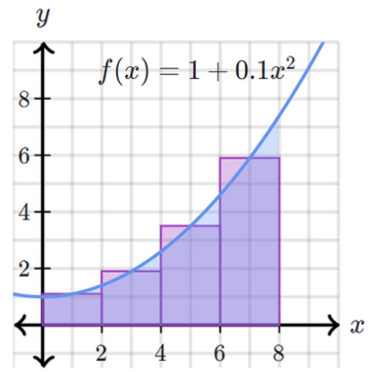
13. Based on your sketches, label the appropriate graph as “most accurate,” “overestimates” or “underestimates.” Compare this to your answers to #11. What is different?

14. Make a conjecture: when a graph is ____________________________, the left Riemann sum (overestimates/underestimates) but when a graph is ____________________________, the left Riemann sum (overestimates/underestimates).

15. Label the type of Riemann sum shown in the following graphs:







a) ____________________________ b) ______________________________ c) _________________________

16. Calculate the Riemann sums of 15 b (above middle):

a) The first rectangle: The base is 2 units.
The height is on the function $f(x) = 1 + 0.1x^2$ where x = 0, or $f(0) = 1 + 0.1 \cdot 0^2 = 1$ unit
The area of rectangle is base x height = $2 \cdot 1 =$ **2 units squared**

b) The second rectangle: The base is 2 units.
The height is $f(2) = 1 + 0.1 \cdot 2^2 = 1.2$ unit
The area of second rectangle = $2 \cdot 1.4 =$ **2.8 units squared**

c) The second rectangle: The base is 2 units.
The height is $f(4) = 1 + 0.1 \cdot 4^2 = 2.6$ unit
The area of third rectangle is _____________________ units squared (fil in the blank)

d) Find the area of the fourth rectangle in 15b.

e) What is the sum of the areas of all four rectangles?

18. Calculate the right Riemann sum for the area under the graph of $f(x) = 1 + 0.1x^2$ on the interval [0 , 8] using 4 equal subintervals? (Use the appropriate graph from #15 above.)

19. Which is an expression for the midpoint Riemann sum for the area under the graph of $f(x) = 1 + 0.1x^2$ on the interval [0 , 8] using 4 equal subintervals?? (Circle one choice).

a) $2 \cdot f(0) + 2 \cdot f(2) + 2 \cdot f(4) + 2f(6)$

b) $2 \cdot f(2) + 2 \cdot f(4) + 2 \cdot f(6) + 2f(8)$

c) $2 \cdot f(1) + 2 \cdot f(3) + 2 \cdot f(5) + 2f(7)$

20. Suppose that an object moving along a straight line path has its velocity in feet per second at time t in seconds given by v(t) = 2 (t ? 3)2 + 2.

(a) Carefully sketch the region whose exact area will tell you the value of the distance the object traveled on the time interval 2 ? t ? 5.

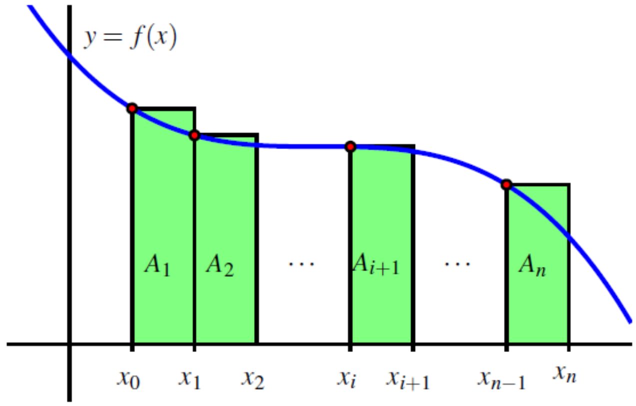
(b) Estimate the distance traveled on [2, 5] by computing L4, R4, and M4.

(c) Does averaging L4 and R4 result in the same value as M4? If not, what do you think the average of L4 and R4 measures?

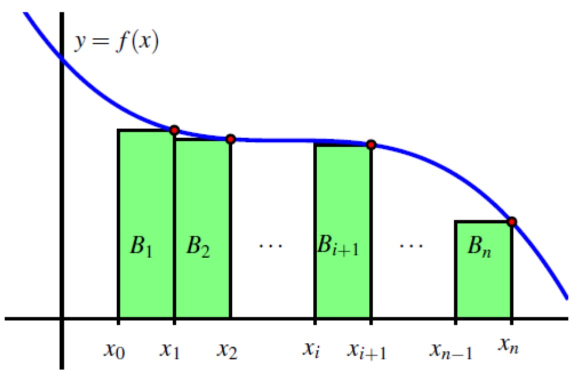
(d) For this question, think about an arbitrary function f , rather than the particular function v given above. If f is positive and increasing on [a, b], will Ln over- estimate or under-estimate the exact area under f on [a, b]? Will Rn over- or under-estimate the exact area under f on [a, b]? Explain.

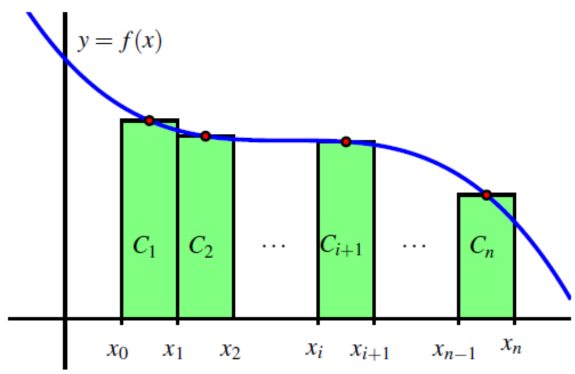
IV. Definition of a Riemann sum

A Riemann sum is simply a sum of products of the form $f(x_{i)} \cdot \Delta x$ that estimates the area between a positive function and the horizontal axis over a given interval. If the function is sometimes negative on the interval, the Riemann sum estimates the difference between the areas that lie above the horizontal axis and those that lie below the axis.



The three most common types of Riemann sums are midpoint sums (shown above) and left or right endpoint sums (shown below) . The only difference among these sums is the location of the point at which the function is evaluated to determine the height of the rectangle whose area is being computed in the sum. For a left Riemann sum, we evaluate the function at the left endpoint of each subinterval, while for right and middle sums, we use right endpoints and midpoints, respectively.





• The left, right, and middle Riemann sums are denoted L_n, R_n, and M_n, with formulas

$$L_n = f(x_0)\Delta x + f(x_1)\Delta x + \cdots + f(x_{n-1})\Delta x = \sum_{i=0}^{n-1} f(x_i)\Delta x,$$

$$R_n = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x = \sum_{i=1}^{n} f(x_i)\Delta x,$$

$$M_n = f(\overline{x}_1)\Delta x + f(\overline{x}_2)\Delta x + \cdots + f(\overline{x}_n)\Delta x = \sum_{i=1}^{n} f(\overline{x}_i)\Delta x,$$

V. Exercises

1. Consider the function f (x) = 3x + 4.

(a) Compute M_4 for $y = f(x)$ on the interval [2, 5]. Be sure to clearly identify the value of $\triangle x$, as well as the locations of $x_0, x_1, \ldots, x_4$. Include a careful sketch of the function and the corresponding rectangles being used in the sum.

(b) Use a familiar geometric formula to determine the exact value of the area of the region bounded by $y = f(x)$ and the x-axis on [2, 5].

(c) Explain why the values you computed in (a) and (b) turn out to be the same. Will this be true if we use a number different than $n = 4$ and compute M_n? Will L_4 or R_4 have the same value as the exact area of the region found in (b)?

(d) Describe the collection of functions g for which it will always be the case that M_n, regardless of the value of n, gives the exact net signed area bounded between the function g and the x-axis on the interval [a, b].

2. Let S be the sum given by

$$S = ((1.4)^2 + 1)\cdot 0.4 + ((1.8)^2 + 1)\cdot 0.4 + ((2.2)^2 + 1)\cdot 0.4 + ((2.6)^2 + 1)\cdot 0.4 + ((3.0)^2 + 1)\cdot 0.4.$$

(a) Assume that S is a right Riemann sum. For what function f and what interval [a, b] is S an approximation of the area under f and above the x-axis on [a, b]? Why?

(b) How does your answer to (a) change if S is a left Riemann sum? a middle Riemann sum?

(c) Suppose that S really is a right Riemann sum. What igeometric quantity does S approximate?

(d) Use sigma notation to write a new sum R that is the right Riemann sum for the same function, but that uses twice as many subintervals as S.

3. A car traveling along a straight road is braking and its velocity is measured at several different points in time, as given in the following table.

seconds, t	0	0.3	0.6	0.9	1.2	1.5	1.8
Velocity in ft/sec, $v(t)$	100	88	74	59	40	19	0

(a) Plot the given data on a set of axes with time on the horizontal axis and the velocity on the vertical axis.

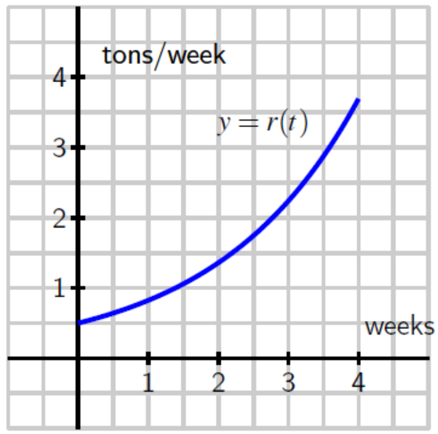
(b) Estimate the total distance traveled during the car the time brakes using a middle Riemann sum with 3 subintervals.

(c) Estimate the total distance traveled on [0, 1.8] by computing L_6, R_6, and

$\frac{1}{2}(L_6 + R_6)$.

(d) Assuming that $v(t)$ is always decreasing on [0, 1.8], what is the maximum possible distance the car traveled before it stopped? Why?

4. The rate at which pollution escapes a scrubbing process at a manufacturing plant increases over time as filters and other technologies become less effective. For this particular example, assume that the rate of pollution (in tons per week) is given by the function r that is pictured below.



(a) Use the graph to estimate the value of M_4 on the interval [0, 4].

(b) What is the meaning of M_4 in terms of the pollution discharged by the plant?

(c) Suppose that $r(t) = 0.5e^{0.5t}$. Use this formula for r to compute L_5 on [0, 4].

(d) Determine an upper bound on the total amount of pollution that can escape the plant during the pictured four week time period that is accurate within an error of at most one ton of pollution.

V. Enduring Understandings, Learning Objectives and Essential Knowledge

Students will know that a Riemann sum, which requires a partition of an interval, I, is the sum of the products, each of which is the value of the function at a point multiplied by the length of the subinterval of the partition. (EK 3.2A1)

I. Investigation:

1. Create a GeoGebra app
2.

II. Review Chapter 3.1-2 in textbook

III. Practice – Khan Academy

1. Complete the first three online practice exercises in the Riemann Sum unit of Khan Academy's AP Calculus AB course: https://www.khanacademy.org/math/ap-calculus-ab/riemann-sums-ab?t=practice

IV. Optional

1. Practice #5 challenge