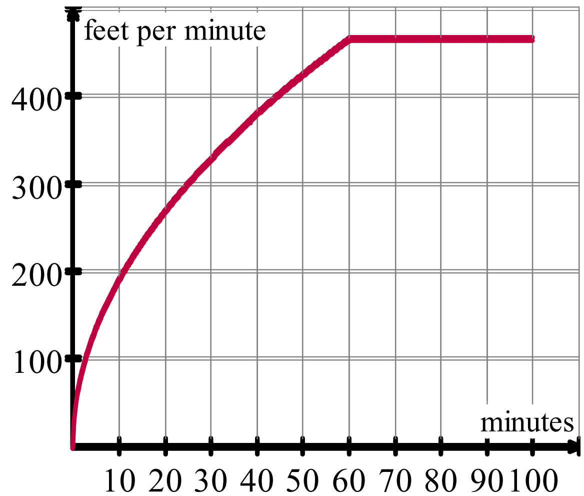
A Jet Tour of Calculus

Day 3: What Can Area Represent?

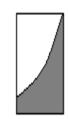
As you pull out on the highway on your road bike you gradually increase your speed according the graph below. Then you notice your speedometer approaching 465 ft per minute so you tap hand brake to slow down your speed to a constant rate of 465 feet per minute.



Notice that the portion of the velocity graph between time t=60 minutes and t=100 minutes is constant. The distance traveled during this time can be represented by what geometric shape? What are the units of measure for the height of region? What are the units of measure for the length (or base) of this region? Using correct units, what is the area of this region? Explain how you determined this unit of measure.

Each rectangular region on the graph represent what distance? Explain how you found your answer.

If the rectangle at the right has the same dimensions at those in the graph above, how many feet does the shaded part of the rectangle represent?



Find an estimate for the distance traveled by the cyclist from t=0 to t = 60 minutes.

Find out how far your bicycle traveled in the first 100 minutes of the trip.

The distance you traveled on your road bike is represented by the bounded area under the velocity graph and above the time axis and the two vertical lines t = 0 and t = 100. This area is called the **definite integral of the velocity** from time t=0 to t= 100 minutes.

You have just found a geometric method to find an approximate value for the definite integral of the velocity from t=0 to t= 60 minutes. It involved both estimating bounded area and using geometric area formulas.

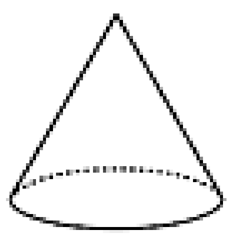
The picture below illustrates a right circular cone sitting on its circular base. Suppose we slice the cone parallel to the base at a point x centimeters from the base. What shape is each cross section?

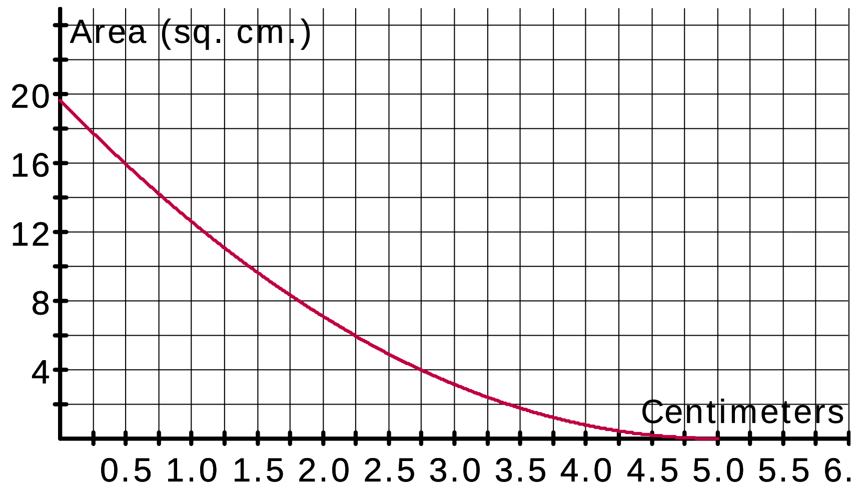
The graph below shows the area of each cross-section as a function of height where we took the cross section.

What is the largest cross-sectional area?

What is the cross-sectional area created at a point 1.75 cm from the base?

At what distance from the base was a 4 square centimeter circle cut?





What does each rectangle in this graph represent? Explain your answer.

The **definite integral of the area** from time x=0 to x=5 centimeters will determine the volume of the cone.

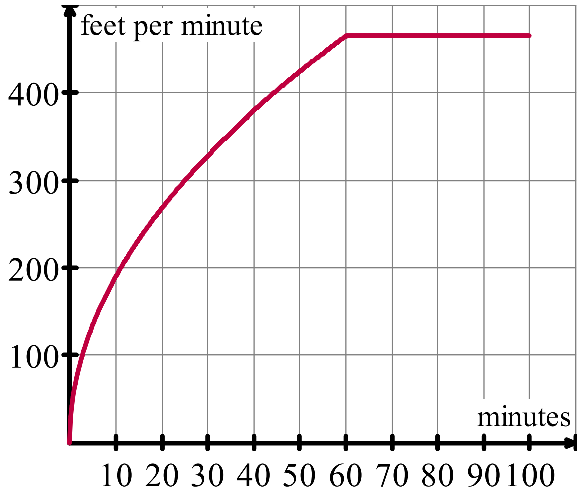
Estimate the definite integral of y, the area, with respect to x for $0 \leq x \leq 5$. Show work that leads to your answer.

A Jet Tour of Calculus

Day 3: What Can Area Represent?

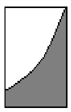
ANSWERS

As you pull out on the highway on your road bike you gradually increase your speed according the graph below. Then you notice your speedometer approaching 465 ft per minute so you tap hand brake to slow down your speed to a constant rate of 465 feet per minute.



Notice that the portion of the velocity graph between time t=60 minutes and t=100 minutes is constant. The distance traveled during this time can be represented by what geometric shape? What are the units of measure for the height of region? What are the units of measure for the length (or base) of this region? Using correct units, what is the area of this region? Explain how you determined this unit of measure. The region is a rectangle. The units for the height are feet per minute. The units for the length are minutes.

Therefore the units for each rectangle are $\frac{\text{feet}}{\text{minute}} \bullet \text{minute} = \text{feet}$.



Each rectangular region on the graph represent what distance? Explain how you found your answer. Each region represent $100\frac{\text{feet}}{\text{minute}} \bullet 10\,\text{minutes} = 1000 \text{ feet}$.

If the rectangle at the right has the same dimensions at those in the graph above, how many feet does the shaded part of the rectangle represent? It appears to represent about 0.4 x 1000 feet or 400 feet.

Find an estimate for the distance traveled by the cyclist from t=0 to t = 60 minutes. about 19,000 feet

Find out how far your bicycle traveled in the first 100 minutes of the trip. About 37,600 feet

The distance you traveled on your road bike is represented by the bounded area under the velocity graph and above the time axis and the two vertical lines t = 0 and t = 100. This area is called the **definite integral of the velocity** from time t=0 to t= 100 minutes.

You have just found a geometric method to find an approximate value for the definite integral of the velocity from t=0 to t= 60 minutes. It involved both estimating bounded area and using geometric area formulas.

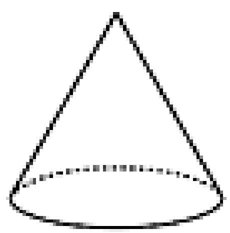
The picture below illustrates a right circular cone sitting on its circular base. Suppose we slice the cone parallel to the base at a point x centimeters from the base. What shape is each cross section? Each shape is a circle.

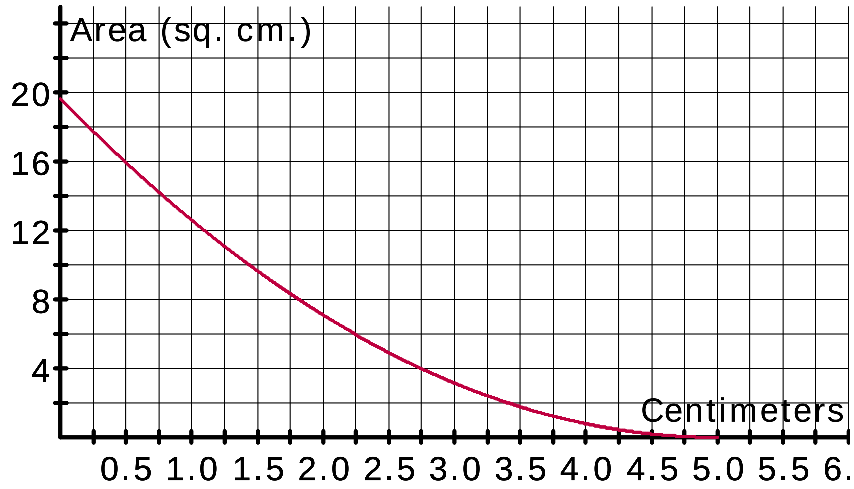
The graph below shows the area of each cross-section as a function of height where we took the cross section.

What is the largest cross-sectional area? The largest area is about 20 square centimeters.

What is the cross-sectional area created at a point 1.75 cm from the base? About 8 square centimeters.

At what distance from the base was a 4 square centimeter circle cut? About 2.75 centimeters.





What does each rectangle in this graph represent? Explain your answer. Each rectangle represents $2\,\text{square centimeters} \cdot 0.25\,\text{centimeters} = 0.50\ \text{cubic centimeters}$.

The **definite integral of the area** from time x=0 to x=5 centimeters will determine the volume of the cone.

Estimate the definite integral of y, the area, with respect to x for $0 \le x \le 5$. Show work that leads to your answer. About 32.5 cubic centimeters (Each square in the graph represents 0.5 cubic centimeters.)

A Jet Tour of Calculus

Day 3: What Can Area Represent?

Assignment

1. A hemisphere is placed on the table so it is sitting on a circular base. Each cross-section, parallel to the base, is also a circle. The graph in Figure 1 represents the area of each cross-section at a distance x units from the base. The definite integral from 0 to 5 represents the volume of the hemisphere by calculating the area bounded by the graph. What does each rectangle in the graph represent? Explain your answer. Estimate the volume of the hemisphere. Include units on your answer.

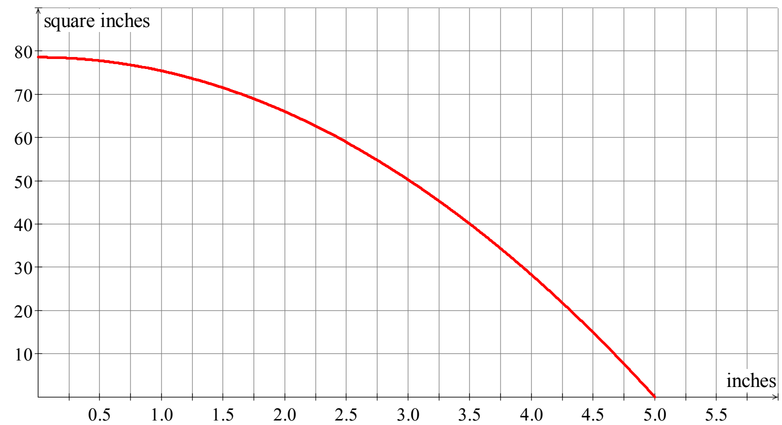


Figure 1
Area of each cross-section of a hemisphere
for $0 \le x \le 5$

2. The velocity of a caterpillar, traveling along a branch, is given in Figure 2. A definite integral from t = 0 to t = 5 would represent the distance the caterpillar travels during the first five seconds. What does each square in the graph represent. Support your reasoning. Estimate the distance the caterpillar travels during the five seconds. If the velocity of the caterpillar is given by $v(t) = \left|32 - \frac{1}{2}t^3\right|$, estimate the rate of change of the velocity (acceleration) of the caterpillar at 2 minutes. Draw a tangent line that represents this rate of change.

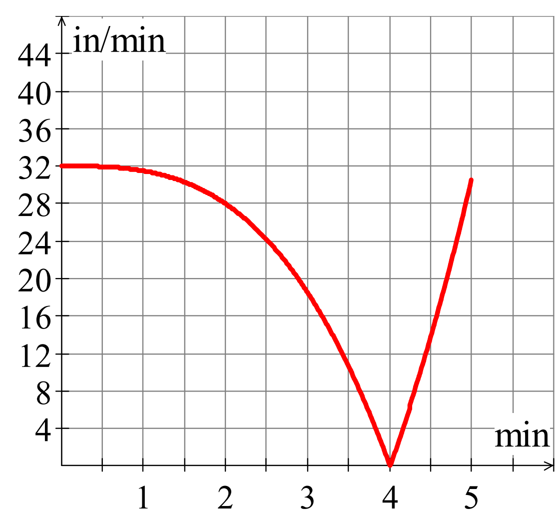


Figure 2
Velocity of a caterpillar

A Jet Tour of Calculus

Day 3: What Can Area Represent?

Assignment Answers

1. A hemisphere is placed on the table so it is sitting on a circular base. Each cross-section, parallel to the base, is also a circle. The graph in Figure 1 represents the area of each cross-section at a distance x units from the base. The definite integral from 0 to 5 represents the volume of the hemisphere by calculating the area bounded by the graph. What does each rectangle in the graph represent? Explain your answer. Estimate the volume of the hemisphere. Include units on your answer.

Each rectangle in the graph represents (10 square inches)(0.25 inches) or 2.5 cubic inches. The hemisphere has a volume of about 260 cubic inches.

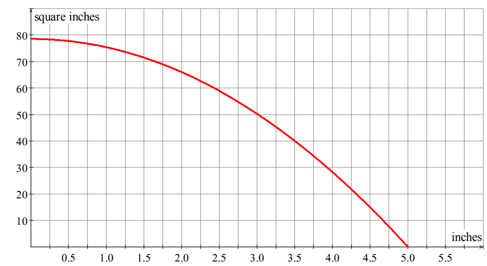


Figure 1
Area of each cross-section of a hemisphere for $0 \le x \le 5$

2. The velocity of a caterpillar, traveling along a branch, is given in Figure 2. A definite integral from t = 0 to t = 5 would represent the distance the caterpillar travels during the first five seconds. What does each square in the graph represent. Support your reasoning. Estimate the distance the caterpillar travels during the five seconds. If the velocity of the caterpillar is given by $v(t) = \left|32 - \frac{1}{2}t^3\right|$, estimate the rate of change of the velocity (acceleration) of the caterpillar at 2 minutes. Draw a tangent line that represents this rate of change. Estimate the rate of change of the velocity (acceleration) of the caterpillar at 2 minutes. Show all work that leads to your answer. Draw a tangent line that represents this rate of change.

Each square in the graph represents (4 in/min)(0.5 min) or 2 inches. At the end of 5 minutes the caterpillar will have travel about 110 inches.

Time Interval	[1.9,2]	[1.99,2]	[1.999,2]
Rate of Change	-5.705	-5.97005	-5.9970005
Time Interval	[1.9999,2]	[1.99999,2]	[1.999999,2]
Rate of Change	-5.9997	-5.99997	-5.999997

The rate of change of the velocity at 2 is about -6 in/min/min.

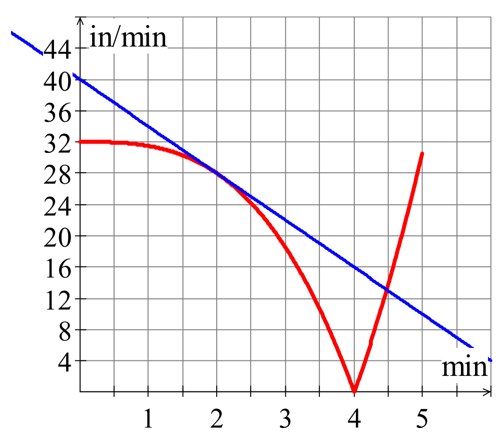


Figure 2
Velocity of a caterpillar

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A Jet Tour of Calculus

Day 4: Determining a Definite Integral with Formulas

1. Water is being pumped into a large storage tank at a rate, $R(t) = (x-2)^3 + 12$ thousands of gallons/day. Draw a sketch of R(t) in Figure 1 for time $0 \le t \le 4\ days$. The definite integral of R(t) from t = 0 to t = 4 represents the thousands of gallons pumped into the tank during the four days.

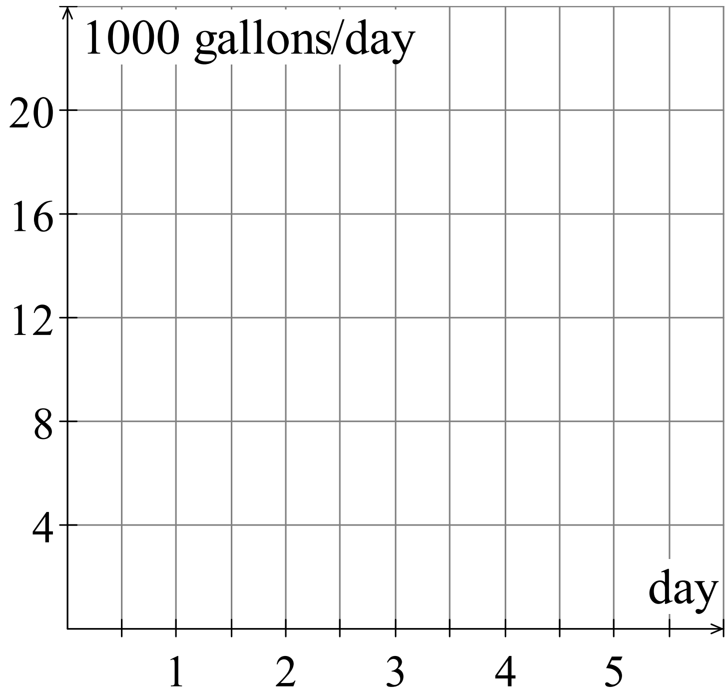


Figure 1

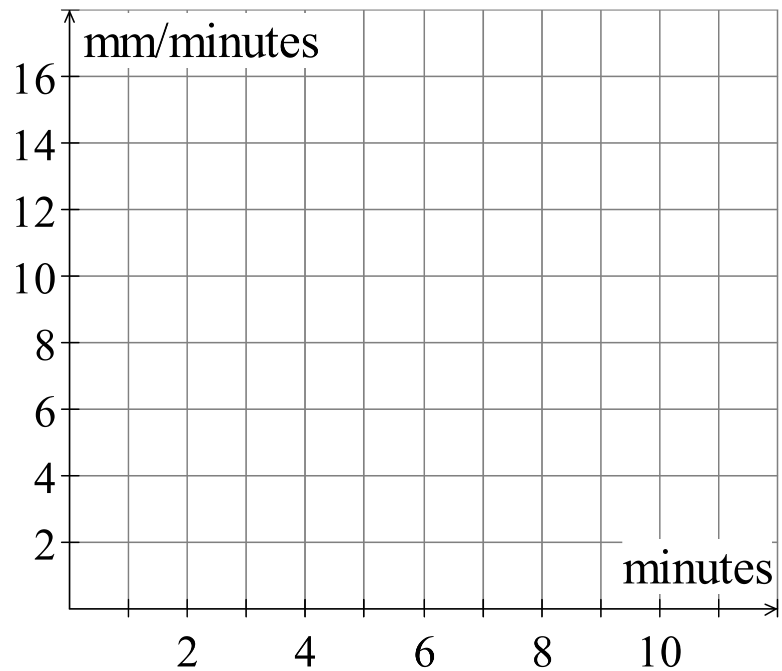
What does the area of each rectangular region represent in this problem?

Draw five vertical line segments to separate the time interval $0 \le t \le 4\ days$ into four equal regions. Use these five line segments to create rectangles that will approximate the area under the graph. Find an estimate for the number of gallons pumped into the tank during the four days.

Draw four additional vertical line segments to separate the time interval $0 \le t \le 4\ days$ into eight equal regions. Find a second estimate for the number of gallons pumped into the tank during the four days.

Explain how increasing the number of line segments will change your estimate for the number of gallons being pumped into the tank.

2. The velocity of an inch worm is given by the function $v(t) = 8x - x^2$. Draw a sketch of this function in figure 2.



What does the area of each rectangle in the graph represent?

Draw vertical line segments to separate the time interval $0 \le t \le 8$ seconds into four equal regions. Use these line segments, and points along the graph of v(t) to create triangles or trapezoids that will approximate the area under the graph. Find an estimate for the distance traveled by the inch worm in 8 seconds.

Draw additional vertical line segments to separate the time interval $0 \le t \le 8$ seconds into more equal regions. Use these new intervals and points along the graph of v(t) to find a second estimate for the distance traveled by the inch worm in 8 seconds.

Explain how additional number of line segments will change your estimate for the distance traveled by the inch worm in 8 seconds.

A Jet Tour of Calculus

Day 4: Determining a Definite Integral with Formulas

ANSWERS

1. Water is being pumped into a large storage tank at a rate, $R(t)=(x-2)^3+12$ thousands of gallons/day. Draw a sketch of R(t) in Figure 1 for time $0 \le t \le 4\ days$. The definite integral of R(t) from t = 0 to t = 4 represents the thousands of gallons pumped into the tank during the four days.

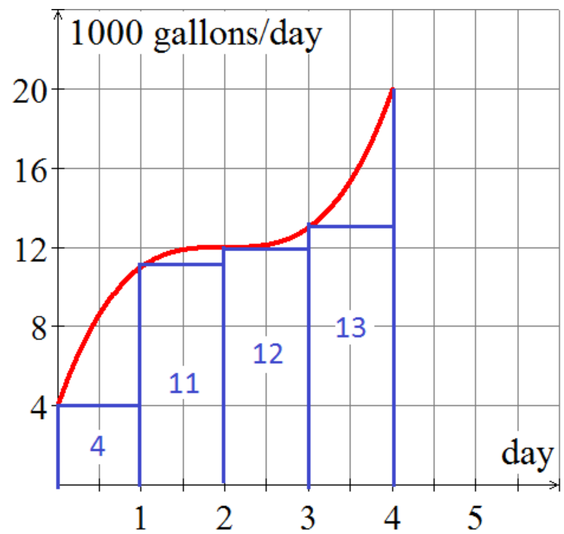
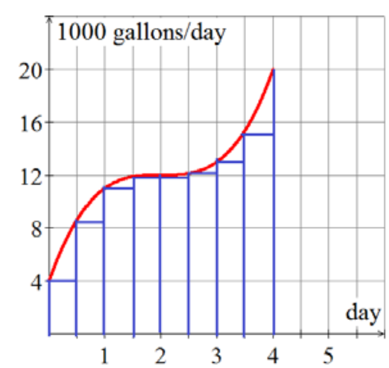


Figure 1

What does the area of each rectangular region represent in this problem? Each rectangle represent 2000 gallons because the dimensions are 4000 gallons/day by 0.5 days.

Draw five vertical line segments to separate the time interval $0 \le t \le 4\ days$ into four equal regions. Use these five line segments to create rectangles that will approximate the area under the graph. Find an estimate for the number of gallons pumped into the tank during the four days.
Answers will vary. The figure shows four rectangles, whose height is drawn at the left hand endpoint of each interval. The numbers in each rectangle are in 1000's of gallons. These rectangles have areas that add up to 40,000 gallons. Students may use other types of rectangles and have answers between 40,000 and 56,000 gallons.



Draw four additional vertical line segments to separate the time interval $0 \le t \le 4\ days$ into eight equal regions. Find a second estimate for the number of gallons pumped into the tank during the four days. Answers will vary. As four additional line segments are added the area can range between 44,000 and 56,000 gallons.

Explain how increasing the number of line segments will change your estimate for the number of gallons being pumped into the tank.

As the number of rectangles are increased the answer for the number of gallons pumped into the tank approach 48,000 gallons, the actual area.

2. The velocity of an inch worm is given by the function $v(t) = 8x - x^2$. Draw a sketch of this function in figure 2.

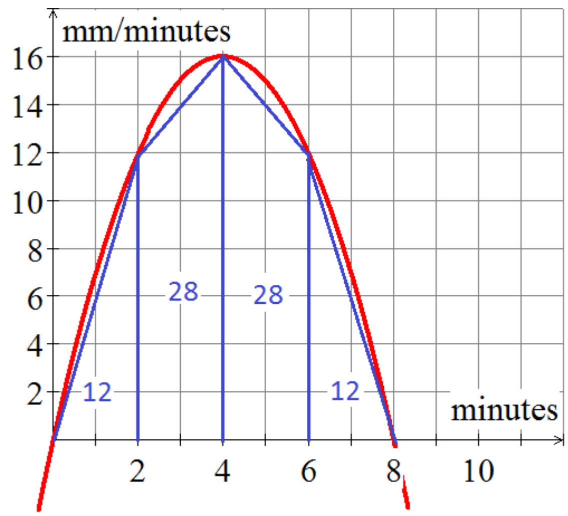
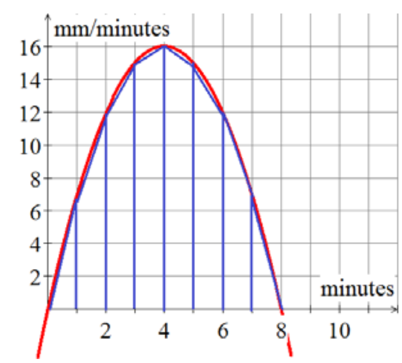


Figure 2

What does the area of each rectangle in the graph represent? The area represents 2 mm because the dimensions are 2 mm/minute by 1 minute.

Draw vertical line segments to separate the time interval $0 \le t \le 8$ seconds into four equal regions. Use these line segments, and points along the graph of v(t) to create triangles or trapezoids that will approximate the area under the graph. Find an estimate for the distance traveled by the inch worm in 8 minutes. The area of the two triangles and two trapezoids adds up to 80 mm. This is how far the inch worm crawls in 8 minutes.

Draw additional vertical line segments to separate the time interval $0 \le t \le 8$ seconds into more equal regions. Use these new intervals and points along the graph of v(t) to find a second estimate for the distance traveled by the inch worm in 8 seconds. Using eight trapezoids and/or triangles the area will be 84 mm.



Explain how additional number of line segments will change your estimate for the distance traveled by the inch worm in 8 seconds. Will these estimates be an over or under estimate for the distance?

As additional trapezoids are added the area of all the triangles and trapezoids will approach the exact area bounded under the velocity graph because the slanted sides better approximate the curvature of the velocity graph. The area will approach the value of 85 1/3 mm. These estimates will be under estimates since the graph of v(t) is concave down. The straight segments will be below the graph

A Jet Tour of Calculus

Day 4: Determining a Definite Integral with Formulas

Assignment

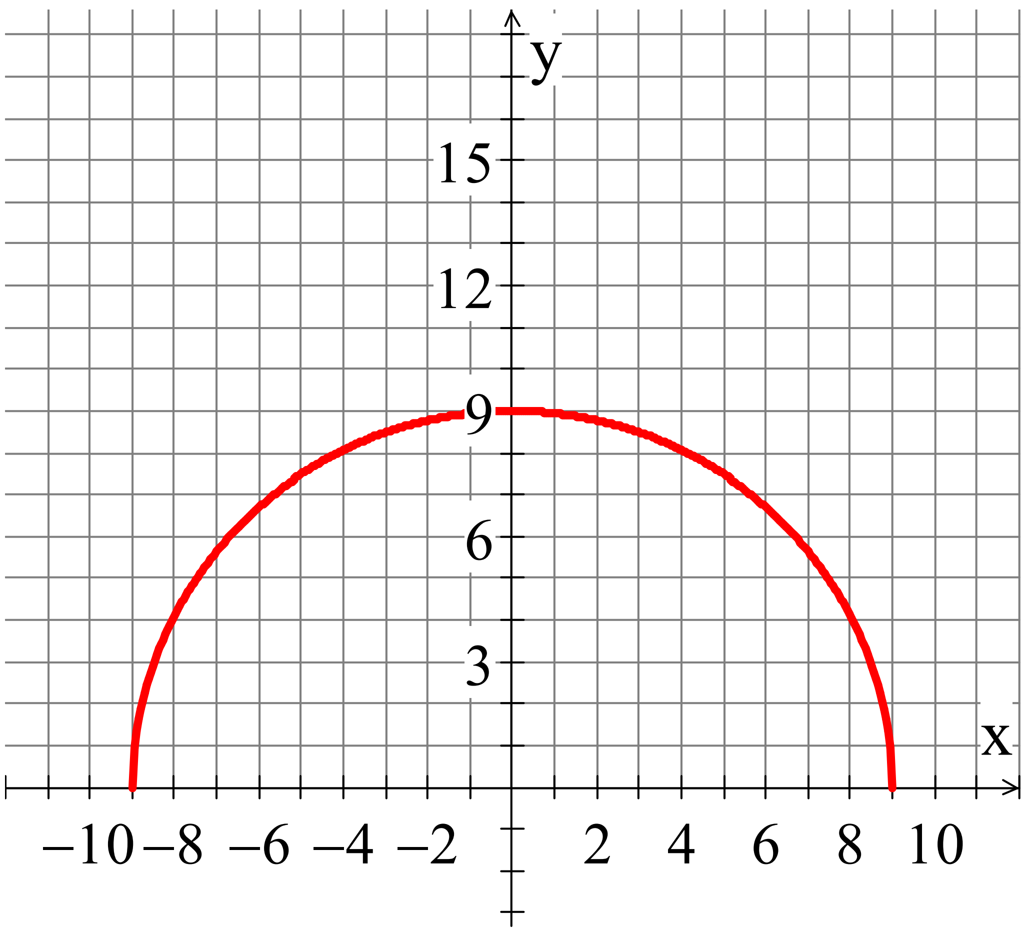
1. A region R is defined by the graph of $f(x)=\sqrt{81-x^2}$ and the x-axis is graphed in Figure 1. What does the area of each of the squares on the graph represent?

   Estimate the bounded area by thinking about the full and partial squares contained in the region R.

   Use 6 rectangles to approximate the area of region R.

   Approximate area of the bounded region R using 6 trapezoids and/or triangles.

   Explain why the three estimates differ from each other. Explain why one of the answer is a better approximation.



2. A solid is sliced into cross sections whose area, $A(x)$, is represented by the graph in Figure 2. The definite integral of $A(x)$ from $x=0$ to $x = 4$ will find the volume of the solid.

What does the area of each of the rectangles on the graph represent in the context of this problem?

Use two different estimation techniques to approximate volume of the solid. Compare the two estimates to the actual volume of the solid.

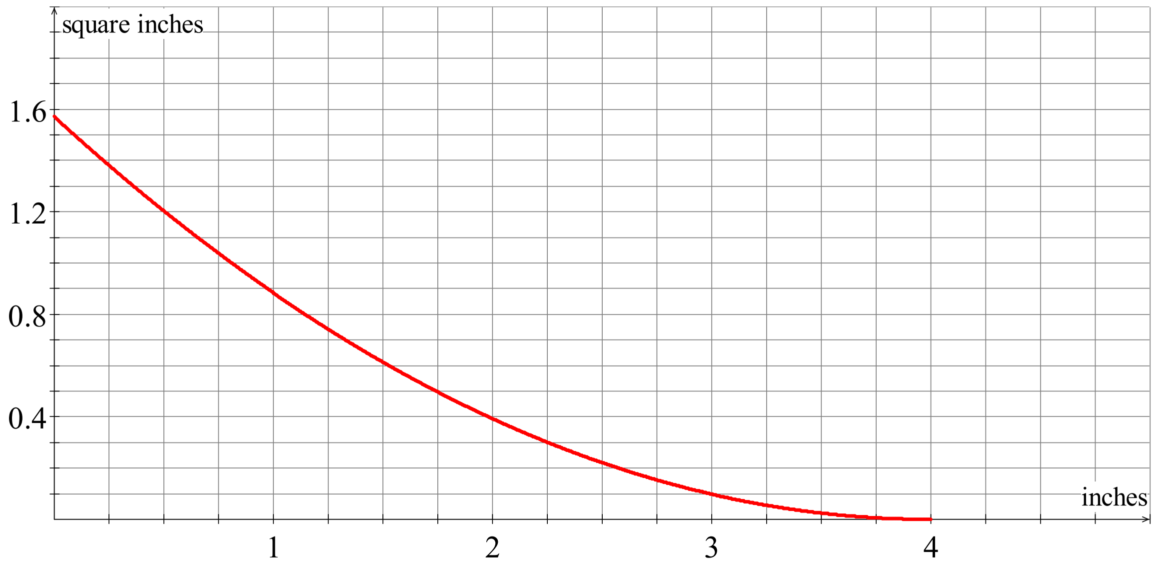


Figure 2

A Jet Tour of Calculus

Day 4: Determining a Definite Integral with Formulas

Assignment Answers

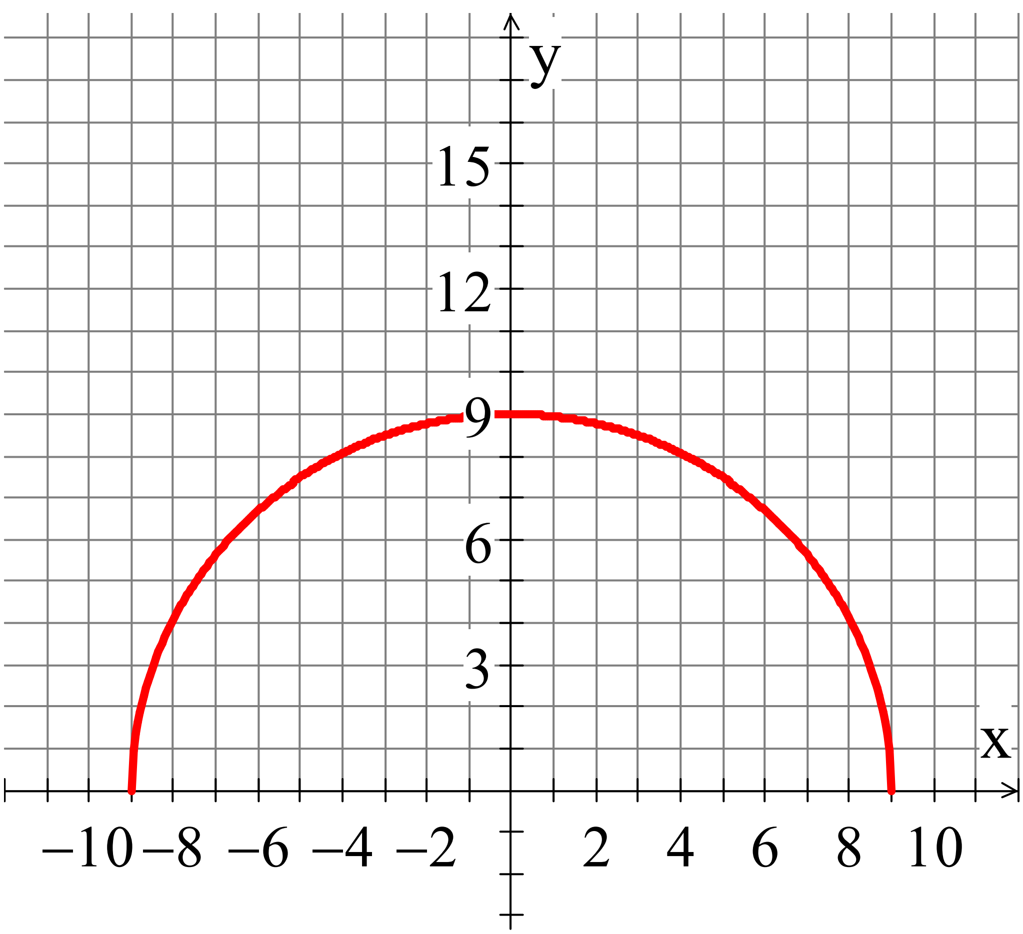
1. A region R is defined by the graph of $f(x) = \sqrt{81 - x^2}$ and the x-axis is graphed in Figure 1. What does the area of each of the squares on the graph represent? Each rectangle represents 1 square unit.

Estimate the bounded area by thinking about the full and partial squares contained in the region R. The area of the bounded region is about 108 full and/or partial squares or about 108 cubic units.

Use 6 rectangles to approximate the area of region R. Using 6 rectangles the area can be between 92 square units and 145 square units.

Approximate area of the bounded region R using 6 trapezoids and/or triangles. Using 6 trapezoids and/or triangles the area will be about 118.16 square units.

Explain why the three estimates differ from each other. Explain why one of the answer is a better approximation. The estimate using trapezoids and/or triangles will be closer to the actual area since the shapes fit closer to the actual shape, but less than the actual area. The actual region is a semicircle with a radius of 9 so its area is $\frac{81}{2}\pi$ or 127.234 square units. The rectangles extend over the graph or under the graph, but the trapezoids fit tighter to the graph and all remain under the graph.



2. A solid is sliced into cross sections whose area, A(x), is represented by the graph in Figure 2. The definite integral of A(x) from x=0 to x = 4 will find the volume of the solid.

What does the area of each of the rectangles on the graph represent in the context of this problem?
Each rectangle represent 0.025 cubic inches of volume since the dimension are 0.1 square inches by 0.25 inches.

Use two different estimation techniques to approximate volume of the solid. Compare the two estimates to the actual volume of the solid.

Using 6 rectangles the approximate volume of the solid will be between 1.59 cubic inches and 2.647 cubic inches. Using 6 trapezoids the approximate volume is 2.123 cubic inches. The estimate using 6 trapezoids should be a better approximation than some of the rectangular approximation since the trapezoids fit the shape better than the rectangles. Rectangles that fit below the graph will give an underestimate. Rectangles that fit above the graph will be an overestimate. Trapezoids will give an overestimater.

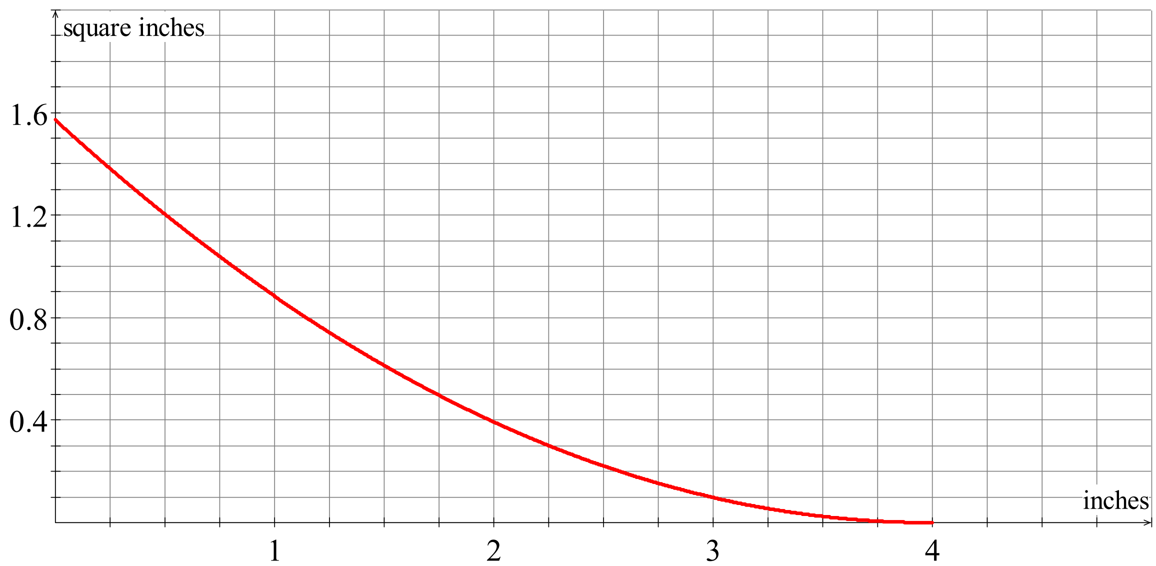


Figure 2