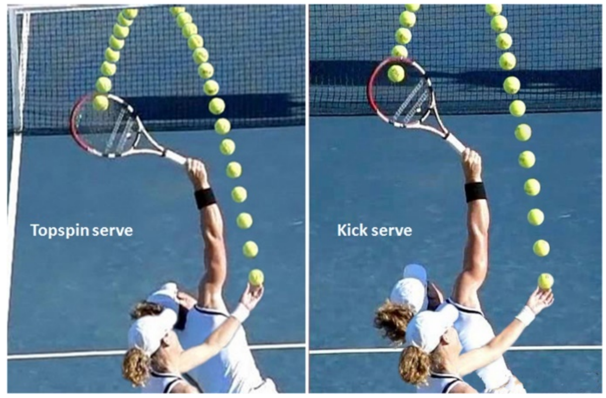
3.1 Can I Get An Inverse?

Intro to Integral Calculus



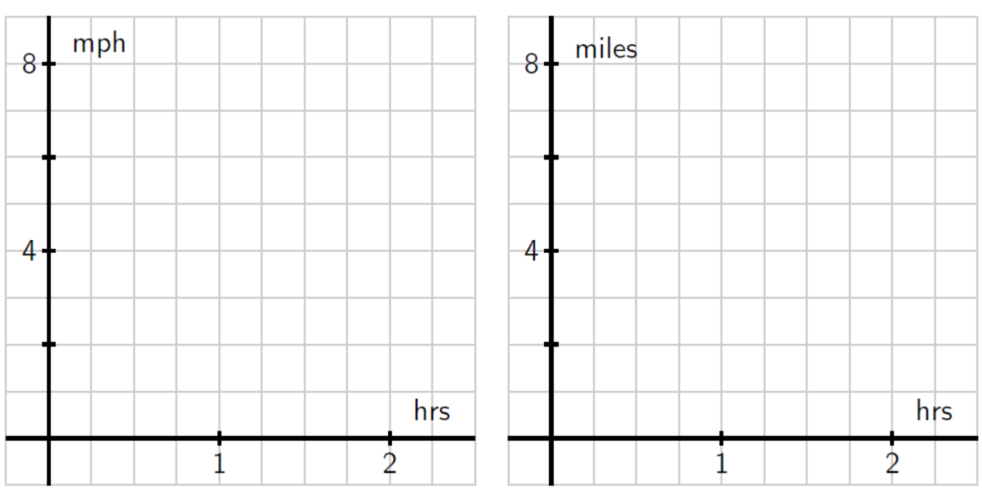
Earlier, you considered a situation where a moving object had a known position at time t. A tennis ball tossed into the air had its height s (in feet) at time t (in seconds) given by $s(t) = 64 - (t - 1)^2$. From this, you investigated the average velocity of the ball on a given interval [a, b], computed by the difference quotient $\frac{s(b)-s(a)}{b-a}$, and found that you could determine the instantaneous velocity of the ball at time t by taking the derivative of the position function,

$$s'(t) = \lim_{h\to 0} \frac{s(t = h) - s(t)}{h}$$

Therefore, given a differentiable position function, you are able to know the exact velocity of the moving object at any point in time. What about the inverse? If you know the instantaneous velocity of an object moving along a straight-line path, can you determine its corresponding position function?

Investigation 3.1.1 - Suppose a person is taking a walk along a long straight path and walks at a constant rate of 3 miles per hour.

1. On the left-hand axes in the figure below, sketch a labeled graph of the velocity function $v(t) = 3$. Note that while the scale on the two sets of axes is the same, the units on the right-hand axes differ from those on the left. The right-hand axes will be used in question 4.



2. How far did the person travel during the two hours? How is this distance related to the area of a certain region under the graph of $y = v(t)$?

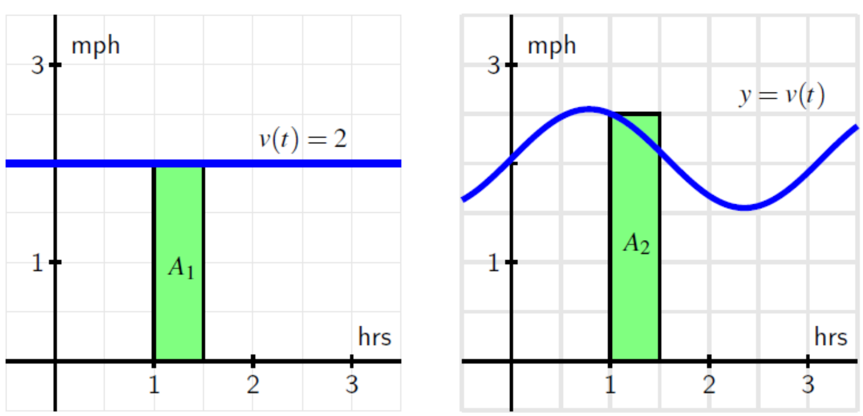
3. Find an algebraic formula, $s(t)$, for the position of the person at time t, assuming that $s(0) = 0$. Explain your thinking.

4. On the right-hand axes, sketch a labeled graph of the position function $y = s(t)$.

5. For what values of t is the position function s increasing? Explain why this is the case using relevant information about the velocity function v.

II. Area under the graph of the velocity function

In Investigation 3.1, you encountered a fundamental fact: when a moving object's velocity is constant (and positive), the area under the velocity curve over a given interval tells us the distance the object traveled. As seen below left, if you consider an object



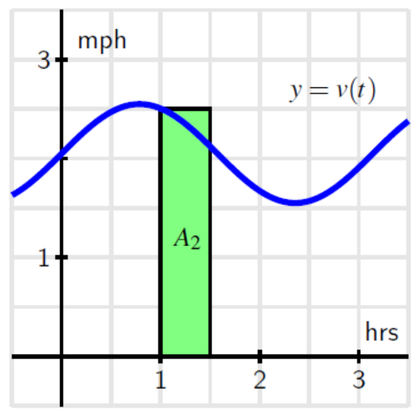
moving at 2 miles per hour over the time interval [1, 1.5], then the area A_1 of the shaded region under $y = v(t)$ on [1, 1.5] is $A_1 = 2\,\frac{\text{miles}}{\text{hour}} \cdot \frac{1}{2}\,\text{hours} = 1\text{ mile}$.

This principle holds in general simply due to the fact that distance equals rate times time, provided the rate is constant. Thus, if $v(t)$ is constant on the interval [a, b], then the distance traveled on [a, b] is the area A that is given by

$$A = v(a)(b - a) = v(a) \triangle t,$$

Where $\triangle t$ is the change in t over the interval. Note, too, that we could use any value of $v(t)$ on the interval [a, b] , since the velocity is constant; we simply chose $v(a)$, the value at the interval's left endpoint.

The situation is obviously more complicated when the velocity function is not constant. At the same time, on relatively small intervals on which $v(t)$ does not vary much, the area principle allows us to estimate the distance the moving object travels on that time interval.



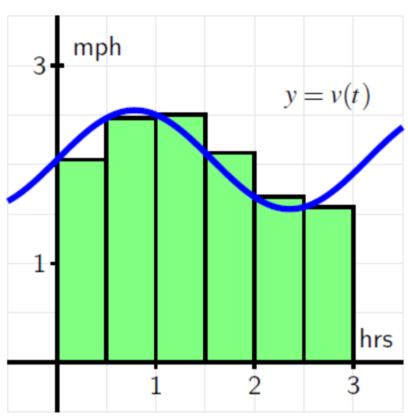
For instance, for the non-constant velocity function shown at right, notice that on the interval [1, 1.5] , velocity varies from $v(1) = 2.5$ to $v(1.5) \approx 2.1$.

Therefore, one estimate for distance traveled is the area of the pictured rectangle,

$A_2 = 2.5 \frac{\text{miles}}{\text{hour}} \cdot \frac{1}{2} \text{hours} = 1.25 \text{ miles}.$

Because v is decreasing on [1, 1.5] and the rectangle lies above the curve, clearly $A_2 = 1.25$ miles is an over-estimate of the actual distance traveled.

If we want to estimate the area under the non-constant velocity function on a wider interval, say [0, 3], it becomes apparent that one rectangle probably will not give a good approximation. Instead, we could use the six rectangles pictured below, find thearea of each rectangle, and add up the total.

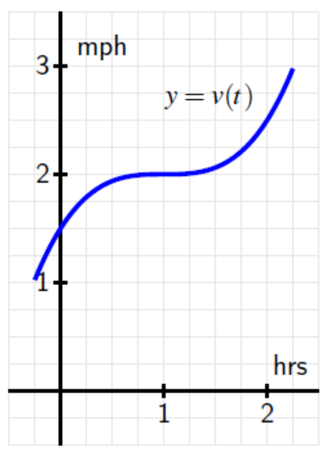


Obviously, there are choices to make and issues to understand: how many rectangles should we use? where should we evaluate the function to decide the rectangle's height? what happens if velocity is sometimes negative? can we attain the exact area under any non-constant curve?

Investigation 3.1.2

Suppose a person is walking in such a way that her velocity varies slightly according to the information given in the table and graph below.

t	0.00	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00
$v(t)$	1.500	1.789	1.938	1.992	2.000	2.008	2.063	2.211	2.500



The graph of $y = v(t)$

6. Using the grid, graph, and given data, estimate the distance traveled by the walker during the two-hour interval from $t = 0$ to $t = 2$. Use time intervals of width $t = 0.5$, in a consistent way.

7. How could you get a better approximation of the distance traveled on 0, 2 ? Explain, and then find this new estimate.

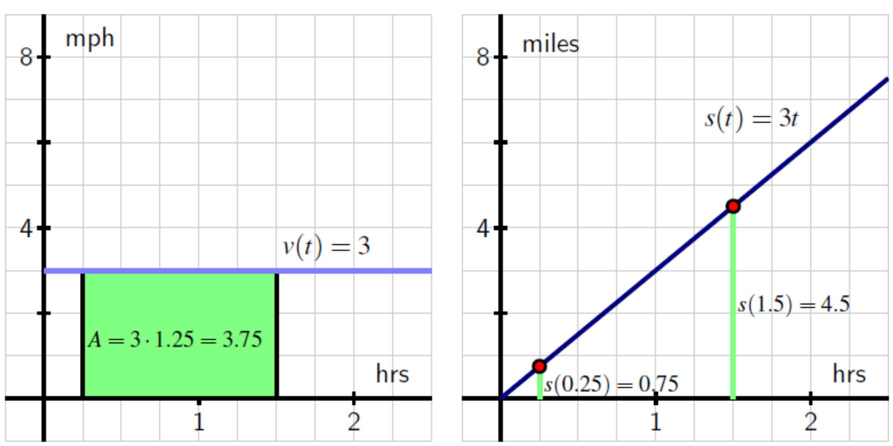
8. Now suppose that you know that *v* is given by $v(t) = 0.5t^3\ 1.5t^2 + 1.5t + 1.5$. Remember that *v* is the **derivative** of the walker's position function, *s*. Find a formula for *s* so that $s' = v$.

9. Based on your work in #8, what is the value of $s(2) - s(0)$? What is the meaning of this quantity?

III. Two approaches: area and antidifferentiation

When the velocity of a moving object is positive, the object's position is always increasing. If you know a formula for the instantaneous velocity, $y = v(t)$, of the moving body at time *t*, then *v* must be the derivative of some corresponding position function *s*. If we can find a formula for *s* from the formula for *v*, it follows that we know the position of the object at time *t*. In addition, under the assumption that velocity is positive, the change in position over a given interval then tells us the distance traveled on that interval.

Consider the situation from Investigation 3.1.1, where a person is walking along a straight line and has velocity function $v(t) = 3$ mph. As pictured below, you can see the already noted relationship between area and distance traveled on the left-hand graph of the velocity function.



In addition, because the velocity is constant at 3, we know that if[1] $s(t) = 3t$, then $s'(t) = 3$,

[1] Here making the implicit assumption that $s(0) = 0$

so $s(t) = 3t$, is a function whose derivative is $v(t)$. Furthermore, observe that $s(1.5) = 4.5$ and $s(0.25) = 0.75$, which are the respective locations of the person at times $t = 0.25$ and $t = 1.5$, and therefore

$$s(1.5) - s(0.25) = 4.5 - 0.75 = 3.75 \text{ miles.}$$

This is not only the change in position on [0.25, 1.5], but also precisely the distance traveled on [0.25, 1.5], which can also be computed by finding the area under the velocity curve over the same interval.

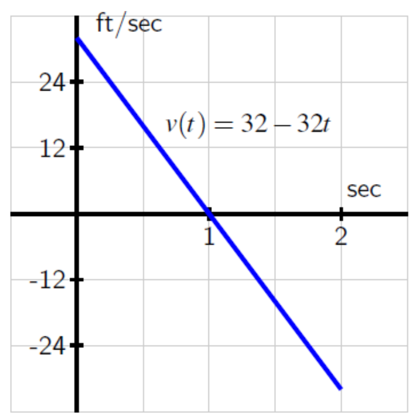
It is most important to observe that if you are given a formula for a velocity function v, it can be very helpful to find a function s that satisfies $s' = v$. In this context, one says that s is an ***antiderivative*** of v. Note that one says "an" antiderivative of f rather than "the" antiderivative of f because it is not the only antiderivative of f.

Investigation 3.1.3

A ball is tossed vertically in such a way that its velocity function is given by $v(t) = 32 - 32t$, where t is measured in seconds and v in feet per second. Assume that this function is valid for $0 \le t \le 2$.

10. For what values of t is the velocity of the ball positive? What does this tell you about the motion of the ball on this interval of time values?

11. Find an antiderivative, s, of v that satisfies $s(0) = 0$.



12. Compute the value of $s(1) - s(1)$. What is the meaning of the value you find?

13. Using the graph of $y = v(t)$ provided at right, find the exact area of the region under the velocity curve between t = 1 and t = 1. What is the meaning of the value you find?

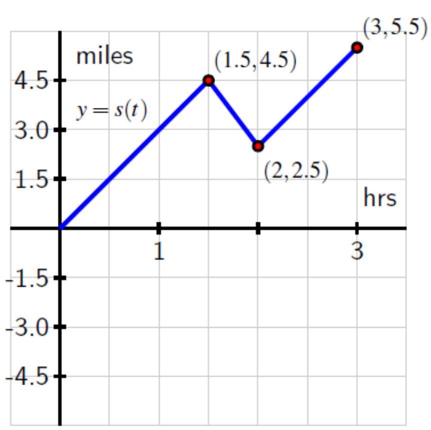
14. Answer the same questions as in 12 and 13 but instead using the interval [0, 1].

15. What is the value of $s(2) - s(0)$? What does this result tell you about the flight of the ball? How is this value connected to the provided graph of $y = v(t)$? Explain.

IV. When velocity is negative

Most of your work in this section has occurred under the assumption that velocity is positive. This hypothesis guarantees that the movement of the object under consideration is always in a single direction, and hence ensures that the moving body's change in position is the same as the distance it travels on a given interval. There are, however, natural settings in which a moving object's velocity is negative; we would like to understand this scenario fully as well.

Consider a simple example where a person goes for a walk on a beach along a stretch of very straight shoreline that runs east-west. Assume that the person's initial position is s $(0) = 0$, and that her position function increases as she moves east from her starting location. For instance, a position of $s = 1$ mile represents being one mile east of the start location, while $s = -1$ tells us the person is one mile west of where they began walking on the beach.



Now suppose the person walks in the following manner. From the outset at $t = 0$, the person walks due east at a constant rate of 3 mph for 1.5 hours. After 1.5 hours, the person stops abruptly and begins walking due west at the constant rate of 4 mph and does so for 0.5 hours. Then, after another abrupt stop and start, the person resumes walking at a constant rate of 3 mph to the east for one more hour. What is the total distance the person traveled on the time interval $t = 0$ to $t = 3$? What is the person's total change in position over that time?

On one hand, these are elementary questions to answer because the velocity involved is constant on each interval. From $t = 0$ to $t = 1.5$, the person traveled

$$D_{[0,1.5]} = 3 \text{ miles per hour} \cdot 1.5 \text{ hours} = 4.5 \text{ miles.}$$

Similarly, on t = 1.5 to t = 2, having a different rate, the distance traveled is

$$D_{[1.5,2]} = 4 \text{ miles per hour} \cdot 0.5 \text{ hours} = 2 \text{ miles.}$$

Finally, similar calculations reveal that in the final hour, the person walked

$$D_{[2,3]} = 3 \text{ miles per hour} \cdot 1 \text{ hours} = 3 \text{ miles.}$$

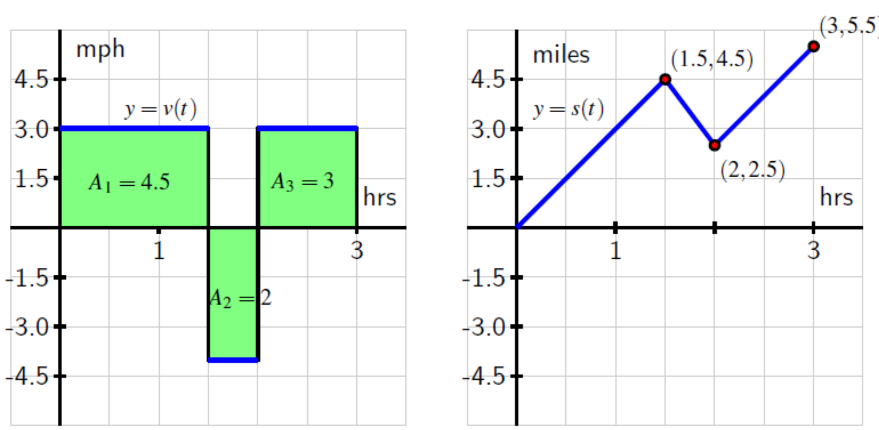
Therefore, the **total** distance traveled is

$$D = D_{[0,1.5]} + D_{[1.5,2]} + D_{[2,3]} = 4.5 + 2 + 3 = 9.5 \text{ miles.}$$

Since the velocity on $1.5 < t < 2$ is actually $v = -4$. Again, the negative sign indicates motion in the westward direction. The calculation above tells you that the person first walked 4.5 miles east, then 2 miles west, followed by 3 more miles east. Therefore, the walker's total change in position is

$$\text{change in position} = 4.5 - 2 + 3 = 5.5 \text{ miles.}$$

In the velocity graph below left, notice how the distances computed above can be viewed as areas: $A_1 = 4.5$ comes from taking rate times time $(3 \cdot 1.5)$, as do A_2 and A_3 for the second and third rectangles.



At left, the velocity function of the person walking; at right, the corresponding position function.

The big new issue is that while A_2 is an area (and is therefore positive), because this area involves an interval on which the velocity function is negative, its area has a negative sign associated with it. This helps us to distinguish between distance traveled and change in position.

The distance traveled is the sum of the areas,

$$D = A_1 + A_2 + A_3 = 4.5 + 2 + 3 = 9.5 \text{ miles.}$$

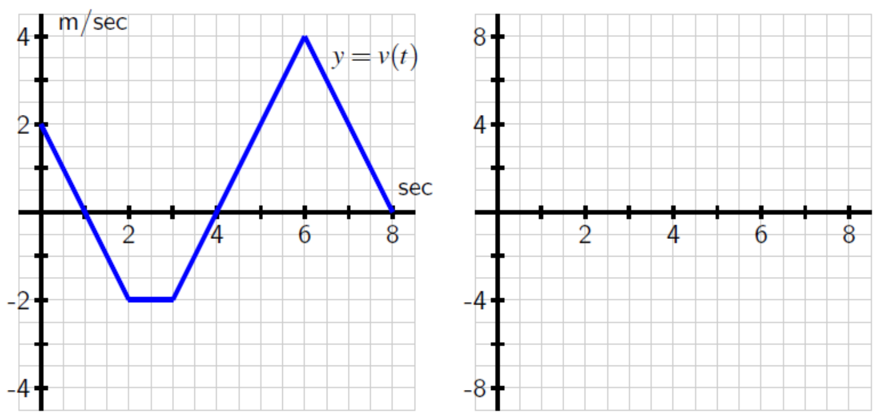
But the change in position must account for the sign associated with the area, where those above the t-axis are considered positive while those below the t-axis are viewed as negative, so that

$$s(3) - s(0) = (+4.5) + (-2) + (+3) = 5.5 \text{ miles,}$$

assigning the "-2" to the area in the interval 1.5, 2 because the velocity is negative and the person is walking in the "negative" direction.

Investigation 3.1.4

Suppose that an object moving along a straight-line path has its velocity v (in meters per second) at time t (in seconds) given by the piecewise linear function whose graph is pictured below. Suppose further that the object's initial position at time $t = 0$ is $s(0) = 1$.

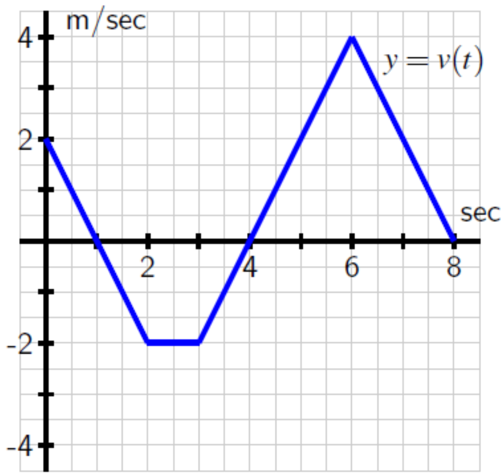


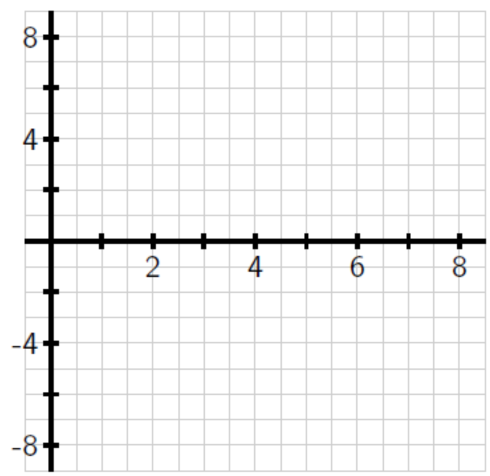
16. Determine the total distance traveled and the total change in position on the time interval $0 \le t \le 2$. What is the object's position at $t = 2$?

17. On what time intervals is the moving object's position function increasing? Why? On what intervals is the object's position decreasing? Why?

18. What is the object's position at $t = 8$? How many total meters has it traveled to get to this point (including distance in both directions)? Is this different from the object's total change in position on $t = 0$ to $t = 8$?

19. Find the exact position of the object at $t = 1, 2, 3, \ldots, 8$ and use this data to sketch an accurate graph of y = s t on the axes provided below right. How can you use the provided information about y = v t to determine the concavity of s on each relevant interval?





V. Enduring Understandings, Learning Objectives and Essential Knowledge

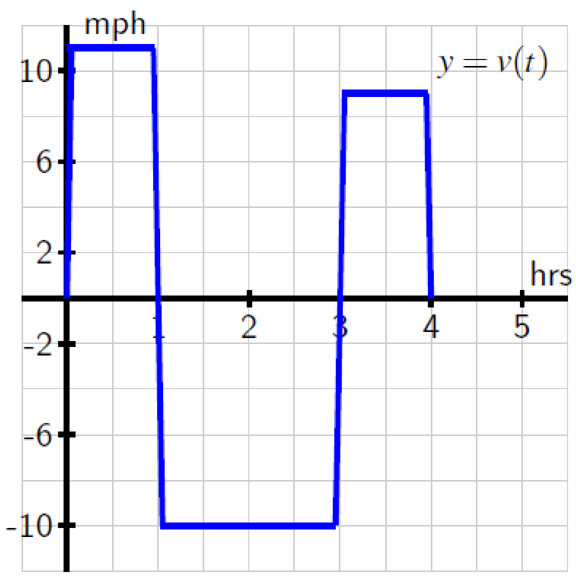
Enduring Understandings EU 3.1: Students will understand that Antidifferentiation is the inverse process of differentiation

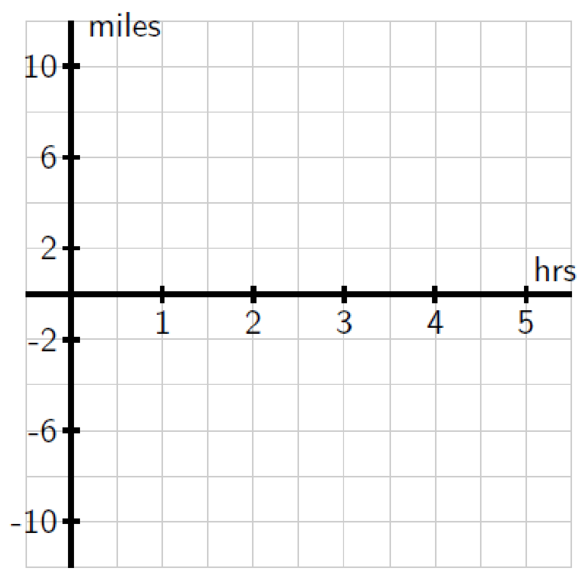
- Students will be able to Recognize antiderivatives of basic functions (LO 3.1A)
    - Students will know that an Antiderivative of a function f is a function g whose derivative is f. (EK 3.1A1)

VI. Practice

1. Along the eastern shore of Lake Michigan from Lake Macatawa (near Holland) to Grand Haven, there is a bike bath that runs almost directly north-south. For the purposes of this problem, assume the road is completely straight, and that the function $s(t)$ tracks the position of the biker along this path in miles north of Pigeon Lake, which lies roughly halfway between the ends of the bike path.

Suppose that the biker's velocity function is given by the graph below on the time interval $0 \le t \le 4$ (where t is measured in hours), and that $s(0) = 1$.





(a) Approximately how far north of Pigeon Lake was the cyclist when she was the greatest distance away from Pigeon Lake? At what time did this occur?

(b) What is the cyclist's total change in position on the time interval $0 \le t \le 2$? At $t = 2$, was she north or south of Pigeon Lake?

(c) What is the total distance the biker traveled on $0 \le t \le 4$? At the end of the ride, how close was she to the point at which she started?

(d) Sketch an approximate graph of $y = s(t)$, the position function of the cyclist, on the interval $0 \le t \le 4$. Label at least four important points on the graph of s.

2. A toy rocket is launched vertically from the ground on a day with no wind. The rocket's vertical velocity at time t (in seconds) is given by $v(t) = 500 - 32t$ feet/sec.

(a) At what time after the rocket is launched does the rocket's velocity equal zero? Call this time value a. What happens to the rocket at $t = a$?

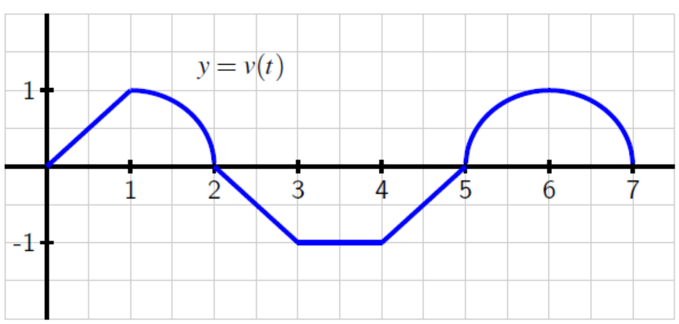
(b) Find the value of the total area enclosed by $y = v(t)$ and the t-axis on the interval $0 \leq t \leq a$. What does this area represent in terms of the physical setting of the problem?

(c) Find an antiderivative s of the function v. That is, find a function s such that $s'(t) = v(t)$.

(d) Compute the value of $s(a) - s(0)$. What does this number represent in terms of the physical setting of the problem?

(e) Compute $s(5) - s(1)$. What does this number tell you about the rocket's flight?

3. An object moving along a horizontal axis has its instantaneous velocity at time t in seconds given by the function v pictured below, where v is measured in feet/sec. Assume that the curves that make up the parts of the graph of $y = v(t)$ are either



(a) Determine the exact total distance the object traveled on $0 \leq t \leq 2$.

(b) What is the value and meaning of $s(5) - s(2)$, where $y = s(t)$ is the position function of the moving object?

(c) On which time interval did the object travel the greatest distance: [0, 2], [2, 4], or [5, 7]?

(d) On which time interval(s) is the position function s increasing? At which point(s) does s achieve a relative maximum?

4. Filters at a water treatment plant become dirtier over time and thus become less effective; they are replaced every 30 days. During one 30-day period, the rate at which pollution passes through the filters into a nearby lake (in units of particulate matter per day) is measured every 6 days and is given in the following table. The time t is measured in days since the filters were replaced.

Day, t	0	6	12	18	24	30
Rate of pollution in units per day p(t)	7	8	10	13	18	35

(a) Plot the given data on a set of axes with time on the horizontal axis and the rate of pollution on the vertical axis.

(b) Explain why the amount of pollution that entered the lake during this 30-day period would be given exactly by the area bounded by $y = p(t)$ and the t-axis on the time interval [0, 30].

(c) Estimate the total amount of pollution entering the lake during this 30-day period. Carefully explain how you determined your estimate.

VII. Assessment – Khan Academy

1. Complete the following online practice exercises from Khan Academy's AP Calculus AB course:
   a. https://www.khanacademy.org/math/ap-calculus-ab/indefinite-integrals-ab/antiderivatives-ab/e/antiderivatives
2.